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Covariance Transformations for Satellite Flight Dynamics Operations*

David A. Vallado**

With the advent of using special perturbations for routine space surveillance operations and the application of covariance information in particular, it's important to understand different representations of the covariance matrix. Recent studies (Chan 2002, Peterson 2002, and others) have focused on the ever increasing role of covariance in determining probability for close-approach calculations. However, these data are often given in a format that's not consistent with every application. For instance, the US Strategic Command (USSTRATCOM) sometimes provides covariance data from numerical differential correction operations in equinoctial elements, while the state vector is in cartesian coordinates. To effectively process these data, transformations must be made between the various formats. Most legacy programs contain FORTRAN code to accomplish these operations, and partial derivatives can be found in some mathematical specifications. This paper documents the transformations necessary to convert between cartesian, equinoctial, and flight (spherical) orbital state formats, as well as various inertial and rotating coordinate systems, and two satellite centered coordinate systems (RSW and NTW, defined later). Detailed equations are given in the appendix including assumptions and limitations (resulting from small eccentricity and inclination values). Both position and velocity vector components are included. A series of test data are presented to show the accuracy of the transformations. For practical implementation, it's also important to understand how various satellite orbits will affect the results of the transformations. Several cases are studied. In addition, how important are the covariance terms in the transformations? Some transformations assume only the diagonal elements are needed for a particular operation. To understand this, some example covariance matrices are converted between formats and propagated into the future.

1. Introduction

The covariance matrix in orbit determination is a usually bi-product of a least squares process (or Kalman filter) and as such, has been present since the first differential corrections were applied to orbital problems. For many decades though, at least in the operational military operations of Air Force Space Command (AFSPC), the covariance matrix was little used and often relegated to a set of software routines that were seldom exercised. During the "reign" of limited analytical theories (SGP4, Hoots 1998 and Vallado, 2001, 651)), the covariance matrix was essentially unavailable***. In contrast, owner operator systems sometimes used covariance information for operations as they had more observational data, precise numerically or detailed analytically (Draper Semianalytical Satellite Theory, Vallado, 2001, 652-658) generated state vectors, better observability, and generally more confidence in the output.

There are numerous applications for the covariance matrix, and as operations in space become more regular, the applications increase. Recent studies (Chan 2002, Peterson 2002, and others) have focused on the increasing role of covariance in determining probability for close-approach calculations. The close-approach problem is a growing concern among satellite operators because the expense of repairing a satellite, or as is more often the case of replacing a satellite after a collision, is prohibitive. Space debris is currently an important topic as there is interest in the piece of debris that separated from the Space Shuttle Columbia before it disintegrated on reentry on February 1, 2003. In the broader sense of space debris, the issue for

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*** It's worth noting that the Aerospace Corporation devised a unique and highly accurate technique (Peterson, et al, 2001) to construct a covariance matrix using just the two-line element set (SGP4) from AFSPC. For applications having access only to two-line elements sets, this technique should definitely be considered.

close-approach becomes one of how to determine whether or not to maneuver if the predictions show a collision. This is especially important for the International Space Station (ISS) as many of its experiments are designed as “zero-g” and maneuvers impart a certain amount of acceleration to the ISS. The universal answer is to apply the covariance matrix and determine the probability of the prediction.

Unfortunately, the covariance is often given in a format that’s not consistent with some applications. For instance, the US Strategic Command (USSTRATCOM) provides some covariance data from special perturbation space surveillance operations in equinoctial elements, while the state vector is in cartesian coordinates*. To effectively process these data, transformations must be made between the two formats. While most legacy programs contain FORTRAN code to accomplish this transformation, the exact partial derivatives are sometimes difficult to locate. This paper documents the transformations necessary to convert between cartesian, equinoctial, and flight (spherical) orbital state formats. In addition, coordinate systems representations are given in inertial (ECI), Mean of Date (MOD), True of Date (TOD), Pseudo Earth Fixed (PEF), Earth Centered Earth Fixed (ECEF), True equator-mean equinox (TEME), Radial-transverse-normal (RSW), and Tangential-velocity-normal (NTW). I do not detail the coordinate system formation, other than identifying their names, because they can be found in Vallado (2001, Ch. 3—a figure on pg 222 relates each). Also note that for this paper, I do not differentiate the International Terrestrial Reference Frame (ITRF) and its different realizations, rather, I group Earth Centered Earth Fixed (ECEF) as a generic group. Likewise, I use ECI to represent the generic inertial coordinate system (FK5, J2000), but not MOD, TOD or TEME. Detailed equations are given including assumptions, limitations (resulting from small eccentricity and inclination values), and some sample test data.

The literature contains a fair amount of information relating to partial derivatives, as well as covariance matrices and their propagation characteristics (Long 1989, Pon 1973, NORAD 1982, TRACE 1977, etc.). In the 1980’s, Wagner (1987) and Douglas (1987) conducted numerous analyses using the covariance matrix and orbit determination methods to apply the results to gravity field determination, orbital selection criteria, and sea-surface determination. Some very useful equations are given by ASTCM (1989), McClain (1992, 79-91), and Cefola and Yurasov (1998) relating the partial derivatives for the direct transformation of equinoctial and cartesian elements. Fraiture (1991) discusses covariance matrices, and emphasizes the application to attitude control systems. He treats the satellite problem briefly, but does not discuss the actual implementation. Thus, I’ve tried to consolidate the information for the specific equations, along with some advice for practical information concerning the implementation of these routines.

Fig. 1 shows various combinations of transformations. Note that there are two types of transformations—those to different coordinate systems, and those to different orbital state formats. From the figure, one may wonder why alternate orbital state formats are not shown for some of the coordinate systems, such as ECEF. This is because when the subsequent transformation is made to classical elements, the classical elements no longer represent the true orbit of the satellite due to the velocity vector changes in the transformation from “inertial” to “fixed” coordinates. Thus, a change to these alternate orbit state formats would not yield useful information.

2. Initial Equations

My first objective was to present detailed equations for the transformation of coordinates. While some element sets are unambiguous, other elements sometimes have multiple definitions. For example, the flight path angle is generally measured from the local horizontal to the velocity vector. However, some centers use the compliment of this angle. Azimuth is generally measured clockwise from North, but is sometimes defined from South. Therefore, I specify the basic equations used to define each coordinate system, and each orbital element set. This will make verifying the resulting transformations and partial derivatives much simpler. Finally, the initial equations also can help identify where singularities can from, such as the eccentricity

*It’s important to note that the equinoctial elements are probably the *best* choice for subsequent propagation of the covariance matrix because their mean element nature makes them act as slow variables, unlike the fast variables in cartesian elements.

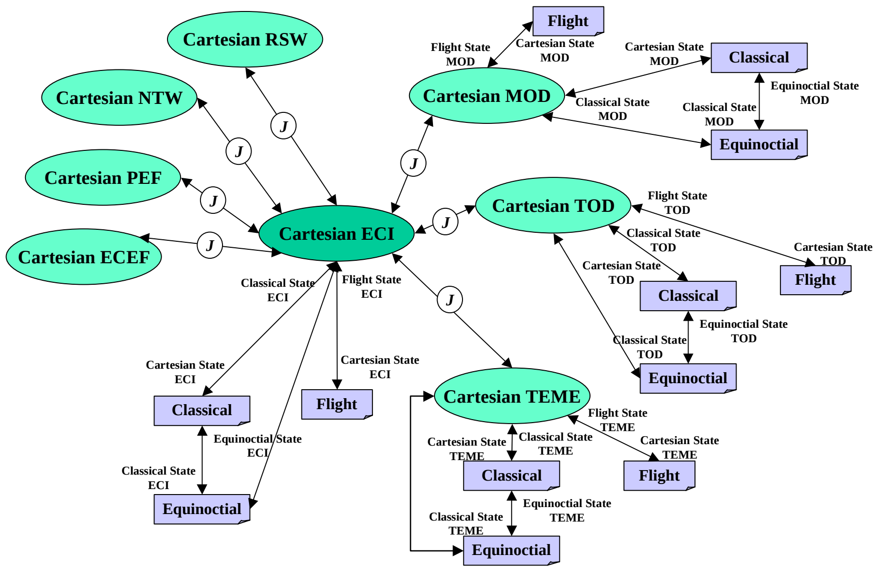


Figure 1. **Covariance Transformations.** This figure shows the types and data required to accomplish each transformation. The central point is the cartesian, ECI covariance matrix. To transform to different coordinate systems, a simple transformation matrix (J) is required. To change the format of the covariance matrix, the state is needed in the desired format (cartesian, classical, etc.).

and inclination values appearing in the denominator. This hinders the transformation's ability to handle certain types of orbits. I've included only a brief description of the coordinate systems as a more detailed description, including figures, is found in Vallado (2001, Ch. 3). The equations for several common orbital state formats are as follows.

2.1 Cartesian Elements

The perifocal coordinate system is defined in Vallado (2001, 161) and it lies in the orbital plane and points to perigee. For cartesian transformations, it's used to relate the classical orbital elements to the position and velocity state vectors.

$$\vec{r}_{PQW} = \begin{bmatrix} \dfrac{p\cos(\nu)}{1+e\cos(\nu)} \\ \dfrac{p\sin(\nu)}{1+e\cos(\nu)} \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v}_{PQW} = \begin{bmatrix} -\sqrt{\dfrac{\mu}{p}}\sin(\nu) \\ \sqrt{\dfrac{\mu}{p}}(e+\cos(\nu)) \\ 0 \end{bmatrix} \tag{1}$$

The transformation from the Geocentric Equatorial Coordinate System (ECI) to the perifocal system is also needed for the partial derivatives (Vallado, 2001, 173)*.

$$\vec{r}_{IJK} = [\text{ROT3}(-\Omega)][\text{ROT1}(-i)][\text{ROT3}(-\omega)]\vec{r}_{PQW} = [\tfrac{IJK}{PQW}]\vec{r}_{PQW}$$

$$\vec{v}_{IJK} = [\text{ROT3}(-\Omega)][\text{ROT1}(-i)][\text{ROT3}(-\omega)]\vec{v}_{PQW} = [\tfrac{IJK}{PQW}]\vec{v}_{PQW}$$

*Note that I use "IJK" to symbolize the axes of the ECI coordinate system.

$$\left[\frac{IJK}{PQW}\right] = \begin{bmatrix} \cos(\Omega)\cos(\omega) - \sin(\Omega)\sin(\omega)\cos(i) & -\cos(\Omega)\sin(\omega) - \sin(\Omega)\cos(\omega)\cos(i) & \sin(\Omega)\sin(i) \\ \sin(\Omega)\cos(\omega) + \cos(\Omega)\sin(\omega)\cos(i) & -\sin(\Omega)\sin(\omega) + \cos(\Omega)\cos(\omega)\cos(i) & -\cos(\Omega)\sin(i) \\ \sin(\omega)\sin(i) & \cos(\omega)\sin(i) & \cos(i) \end{bmatrix}$$

2.2 Classical Elements

Classical elements are determined from position and velocity vectors using the following equations. First, the energy equation is solved for the semi major axis to find

$$a = \frac{\mu r}{-v^2 r + 2\mu}$$

The remaining equations are found following standard practice (Vallado, 2001, 120-121). Notice the multiple possibilities for some of the relations for inclination (i) and right ascension of the ascending node (Ω).

$$\vec{e} = \frac{\left(v^2 - \frac{\mu}{r}\right)\vec{r} - (\vec{r}\cdot\vec{v})\vec{v}}{\mu}$$

$$\vec{h} = \vec{r}\times\vec{v} \qquad h = \left|\vec{h}\right|$$

$$\vec{n} = \hat{K}\times\vec{h}$$

$$\cos(i) = \frac{h_K}{\left|\vec{h}\right|} \qquad \sin(i) = \frac{r_K}{|\vec{r}|} \tag{2}$$

$$\cos(\Omega) = \frac{n_I}{|\vec{n}|} \qquad \cos(\Omega) = \frac{-h_J}{\sqrt{h_I^2 + h_J^2}} \qquad \tan(\Omega) = \frac{h_I}{-h_J}$$

$$\cos(\omega) = \frac{\vec{n}\cdot\vec{e}}{|\vec{n}||\vec{e}|}$$

$$\cos(\nu) = \frac{\vec{e}\cdot\vec{r}}{|\vec{e}||\vec{r}|}$$

$$M = E - e\sin(E) = n\Delta\tau = \sqrt{\frac{\mu}{a^3}}\Delta\tau$$

There are several choices for the sixth element, either true anomaly (ν), mean anomaly (M), Eccentric anomaly (E), or time since perigee passage ($\Delta\tau$). A forward difference approach may be used to go between true and mean anomaly values. Use the following values.

$$\sin(E) = \frac{\sin(\nu)\sqrt{1-e^2}}{1 + e\cos(\nu)} \qquad \text{and} \qquad \sin(\nu) = \frac{\sin(E)\sqrt{1-e^2}}{1 - e\cos(E)}$$

$$\cos(E) = \frac{e + \cos(\nu)}{1 + e\cos(\nu)} \qquad \cos(\nu) = \frac{\cos(E) - e}{1 - e\cos(E)}$$

2.3 Equinoctial Elements

Equinoctial elements (Broucke and Cefola, 1972) are popular because they do not suffer from the singularity problems that classical and other elements do. There are numerous "standard" notations for the element symbols, but I've chosen the set shown below (in addition to some other common symbols). For completely general applications, a multiplier (f_r) is used with the equinoctial elements to specify the direction for exact retrograde orbits (i = 180°). As such, the f_r multiplier is generally included for completeness with equinoctial elements, however, for our purposes we will assume it is always unity, and hence restrict ourselves from processing exact retrograde equatorial orbits. Because there are no exact retrograde orbits, this is a reasonable assumption.

$$
\begin{aligned}
a_f &= k_e = e\,\mathrm{COS}(\omega + \Omega) \\
a_g &= h_e = e\,\mathrm{SIN}(\omega + \Omega) \\
n &= \sqrt{\frac{\mu}{a^3}} \\
L &= \lambda_M = \Omega + \omega + M = \Omega + \omega + n\Delta t \qquad M = E - e\,\mathrm{SIN}(E) \\
\chi &= p_e = \mathrm{TAN}\left(\frac{i}{2}\right)\mathrm{SIN}(\Omega) \\
\psi &= q_e = \mathrm{TAN}\left(\frac{i}{2}\right)\mathrm{COS}(\Omega)
\end{aligned}
\tag{3}
$$

Notice again that there are choices to make when choosing a form of the equations to use in partial derivative operations. The reverse process uses the following relations.

$$
\begin{aligned}
a &= \left(\frac{\mu}{n^2}\right)^{1/3} \\
e &= \sqrt{a_f^2 + a_g^2} \\
i &= 2\,\mathrm{TAN}^{-1}\sqrt{\chi^2 + \psi^2} \\
\Omega &= \mathrm{TAN}^{-1}\left(\frac{\chi}{\psi}\right) \\
\omega &= \mathrm{TAN}^{-1}\left(\frac{a_g}{a_f}\right) - \mathrm{TAN}^{-1}\left(\frac{\chi}{\psi}\right) \\
M &= \lambda_M = L - \omega - \Omega
\end{aligned}
\tag{4}
$$

The direct formulation of cartesian and equinoctial elements may be found in Cefola (1972), Long et al (1989, 3-65 to 3-68), ASTCM (1989, 20-90 to 20-102), McClain (1992), and Cefola and Yurasov (1998). Time did not permit inclusion of all the relevant equations.

2.4 Flight (Spherical) Elements

Because there is some variability in the literature over what constitutes flight parameters, I'll specify them as geocentric latitude (ϕ_{gc}), longitude (λ), flight-path angle from the local horizontal (ϕ_{fpa}), azimuth (β), position magnitude, and velocity magnitude. Remember that when you use Earth fixed parameters (latitude and longitude), the state vectors must be in Earth-fixed (ECEF) coordinates. Spherical parameters use the right ascension (α) and declination (δ) instead of longitude and latitude, for which the change of coordinate system is not necessary. See also Long et al. (1989, 3-42 to 3-44), and O'Conner (1983, 1-29).

$$
\begin{aligned}
\phi_{gc} &= \mathrm{SIN}^{-1}\left(\frac{r_K}{\sqrt{r_I^2 + r_J^2 + r_K^2}}\right) \\
\lambda &= \mathrm{TAN}^{-1}\left(\frac{r_J}{r_I}\right) \\
r &= \sqrt{r_I^2 + r_J^2 + r_K^2}
\end{aligned}
\tag{5}
$$

$$v = \sqrt{v_I^2 + v_J^2 + v_K^2}$$

$$\phi_{fpa} = 90° - \text{TAN}^{-1}\left(\frac{h}{\vec{r} \cdot \vec{v}}\right)$$

$$\vec{A} = \vec{h} \times \vec{r}$$

$$\beta = \text{TAN}^{-1}\left(\frac{r_I A_J - r_J A_I}{A_K r}\right)$$

The azimuth may also be calculated using topocentric coordinate system (SEZ), although this will result in longer partial derivatives (Vallado, 2001, 250-254). In the following equations, the Earth Orientation Parameters (EOP) are used (ΔUT1, ΔAT, x_p, y_p). These represent the UT1 - UTC value, the number of accumulated leap seconds, and the polar motion values, respectively. ***FK5*** and ***SITE*** are simply algorithms from Vallado (2001, 222, 408).

$$\textbf{\textit{FK5}}(\vec{r}_{ECI}, \vec{v}_{ECI}, yr, mo, day, UTC, \Delta UT1, \Delta AT, x_p, y_p \Rightarrow \vec{r}_{ECEF}, \vec{v}_{ECEF})$$

$$\textbf{\textit{SITE}}(\phi_{gd}, \lambda, h_{ellp} \Rightarrow \vec{r}_{siteECEF})$$

$$\vec{\rho}_{ECEF} = \vec{r}_{ECEF} - \vec{r}_{SiteECEF}$$

$$\dot{\vec{\rho}}_{ECEF} = \vec{v}_{ECEF}$$

$$\vec{\rho}_{SEZ} = [\text{ROT}2(90° - \phi_{gd})][\text{ROT}3(\lambda)]\vec{\rho}_{ECEF}$$

$$\dot{\vec{\rho}}_{SEZ} = \underbrace{[\text{ROT}2(90° - \phi_{gd})][\text{ROT}3(\lambda)]}\,\dot{\vec{\rho}}_{ECEF}$$

$$\left[\frac{\text{SEZ}}{\text{ECEF}}\right] = \begin{bmatrix} \text{SIN}(\phi_{gd})\text{COS}(\lambda) & \text{SIN}(\phi_{gd})\text{SIN}(\lambda) & -\text{COS}(\phi_{gd}) \\ -\text{SIN}(\lambda) & \text{COS}(\lambda) & 0 \\ \text{COS}(\phi_{gd})\text{COS}(\lambda) & \text{COS}(\phi_{gd})\text{SIN}(\lambda) & \text{SIN}(\phi_{gd}) \end{bmatrix}$$

$$\text{SIN}(el) = \frac{\rho_Z}{\rho}$$

IF Elevation $\neq 90°$

$$\text{SIN}(\beta) = \frac{\rho_E}{\sqrt{\rho_S^2 + \rho_E^2}} \qquad \text{COS}(\beta) = \frac{-\rho_S}{\sqrt{\rho_S^2 + \rho_E^2}}$$

IF Elevation = $90°$

$$\text{SIN}(\beta) = \frac{\dot{\rho}_E}{\sqrt{\dot{\rho}_S^2 + \dot{\rho}_E^2}} \qquad \text{COS}(\beta) = \frac{-\dot{\rho}_S}{\sqrt{\dot{\rho}_S^2 + \dot{\rho}_E^2}}$$

The reverse process requires first finding the ECEF position vector.

$$\vec{r}_{ECEF} = \begin{bmatrix} r\,\text{COS}(\phi_{gc})\text{COS}(\lambda) \\ r\,\text{COS}(\phi_{gc})\text{SIN}(\lambda) \\ r\,\text{SIN}(\phi_{gc}) \end{bmatrix}$$

Remember to convert the vectors back to inertial if the spherical parameter set (α, δ) is used.

$$\vec{r}_{ECEF}, yr, mo, day, UTC, \Delta UT1, \Delta AT, x_p, y_p \Rightarrow \vec{r}_{ECI}$$

$$\text{SIN}(\delta) = \frac{r_K}{r} \qquad \text{COS}(\delta) = \frac{\sqrt{r_I^2 + r_J^2}}{r} \tag{6}$$

$$\text{SIN}(\alpha) = \frac{r_J}{\sqrt{r_I^2 + r_J^2}} \qquad \text{COS}(\alpha) = \frac{r_I}{\sqrt{r_I^2 + r_J^2}}$$

To find the velocity vector at this point, you must modify the velocity equation in inertial coordinates (Vallado, 2001, 247)

$$\vec{v}_{ECI} = \begin{bmatrix} \dot{r}\,\text{COS}(\delta)\text{COS}(\alpha) - r\,\text{SIN}(\delta)\text{COS}(\alpha)\dot{\delta} - r\,\text{COS}(\delta)\text{SIN}(\alpha)\dot{\alpha} \\ \dot{r}\,\text{COS}(\delta)\text{SIN}(\alpha) - r\,\text{SIN}(\delta)\text{SIN}(\alpha)\dot{\delta} + r\,\text{COS}(\delta)\text{COS}(\alpha)\dot{\alpha} \\ \dot{r}\,\text{SIN}(\delta) + r\,\text{COS}(\delta)\dot{\delta} \end{bmatrix}$$

Using the flight-path angle from the vertical (ϕ_{fpav}) and the azimuth (β),

$$\vec{v}_{ECI} = \begin{bmatrix} v\Big(\text{COS}(\alpha)\big(-\text{COS}(\alpha)\,\text{COS}(\beta)\,\text{SIN}(\phi_{fpav})\,\text{SIN}(\delta) + \text{COS}(\phi_{fpav})\,\text{COS}(\delta)\big) - \text{SIN}(\beta)\,\text{SIN}(\phi_{fpav})\,\text{SIN}(\alpha)\Big) \\ v\Big(\text{SIN}(\alpha)\big(-\text{COS}(\alpha)\,\text{COS}(\beta)\,\text{SIN}(\phi_{fpav})\,\text{SIN}(\delta) + \text{COS}(\phi_{fpav})\,\text{COS}(\delta)\big) + \text{SIN}(\beta)\,\text{SIN}(\phi_{fpav})\,\text{COS}(\alpha)\Big) \\ v\big(\text{COS}(\alpha)\,\text{COS}(\delta)\,\text{SIN}(\phi_{fpav}) + \text{COS}(\phi_{fpav})\,\text{SIN}(\delta)\big) \end{bmatrix}$$

2.5 Satellite Coordinate Systems

Orbital plane coordinate systems often provide useful information for satellite planners. The terminology is usually generic, however I distinguish two satellite coordinate systems, each of which lie in the orbital plane with the third axis normal to that plane (*W* axis). The RSW system moves with the satellite and is sometimes given the letters RTN (radial, transverse, and normal). The *R* axis always points from the Earth's center along the radius vector toward the satellite as it moves through the orbit. The *S* axis points in the direction of (but not necessarily parallel to) the velocity vector and is *perpendicular* to the radius vector—an important distinction. Figure 2 shows this system. The *S* axis is usually not aligned with the velocity vector except for circular orbits or for elliptical orbits at apogee and perigee. In addition, because the coordinate system is based on the satellite's present location, it applies to all orbit types. ***Radial*** positions and displacements are parallel to the position vector (along the *R* axis). ***Along-track*** or ***transverse*** displacements are normal to the position vector (along the *S* axis). Some confusion may exist with *in-track* displacements and the NTW coordinate system which is discussed next. Finally, ***cross-track*** positions are normal to the plane defined by the current position and velocity vectors (along the *W* axis). These orientations mainly provide reference points for describing the satellite's position from a sensor site and are often used to describe orbital errors, relative positions, and displacements of satellite orbits. Given a state consisting of a position and velocity vector, the unit vectors and transformation for this coordinate system are as follows:

$$\hat{R} = \frac{\vec{r}}{|\vec{r}|} \qquad \hat{W} = \frac{\vec{r} \times \vec{v}}{|\vec{r} \times \vec{v}|} \qquad \hat{S} = \hat{W} \times \hat{R} \tag{7}$$

$$\vec{r}_{IJK} = \left[\hat{R} \;\vdots\; \hat{S} \;\vdots\; \hat{W}\right] \vec{r}_{RSW}$$

In the NTW system, the *T* axis is tangential to the orbit and always points to the velocity vector. The *N* axis lies in the orbital plane, normal to the velocity vector, and the *W* axis is normal to the orbital plane (as in

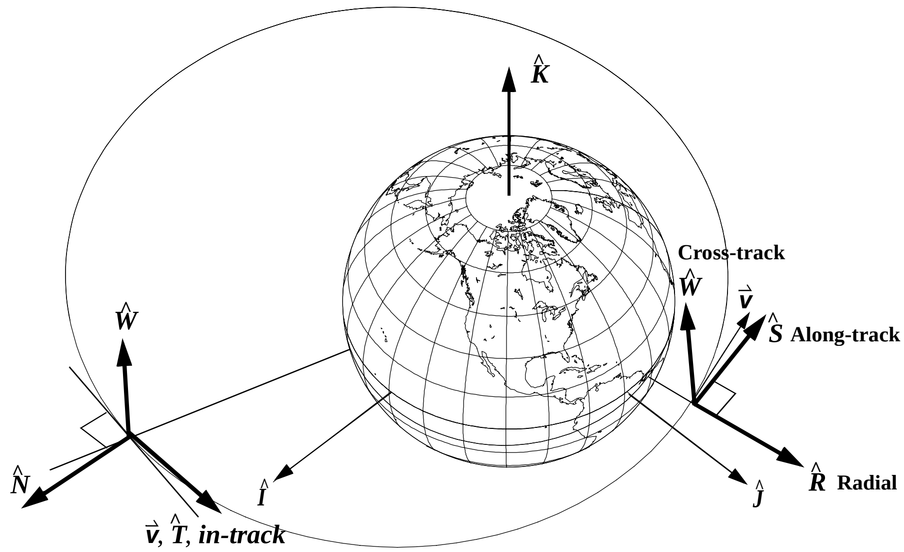


Figure 2. **Satellite Coordinate Systems, RSW and NTW.** These coordinate systems move with the satellite. The *R* axis points to the satellite, the *W* axis is normal to the orbital plane (and *not* usually aligned with the *K* axis), and the *S* axis is normal to the position vector and positive in the direction of the velocity vector. The *S* axis is aligned with the velocity vector *only* for circular orbits. In the NTW system, the *T* axis is always parallel to the velocity vector. The *N* axis is normal to the velocity vector and is *not* aligned with the radius vector, except for circular orbits, and at apogee and perigee in elliptical orbits.

the RSW system). We define ***in-track*** or ***tangential*** displacements as deviations along the *T* axis. In-track errors are *not* the same as along-track variations in the RSW system. One way to remember the distinction is that the in-track errors are *in* the direction of the velocity, whereas along-track variations are simply *along* the velocity vector. We use this coordinate system to analyze drag effects on the orbit because drag always acts along the relative velocity vector. Depending on the attitude, this system is also useful for solar radiation pressure analyses. The NTW coordinate system has the following unit vectors and transformation.

$$\hat{T} = \frac{\vec{v}}{|\vec{v}|} \qquad \hat{W} = \frac{\vec{r}\times\vec{v}}{|\vec{r}\times\vec{v}|} \qquad \hat{N} = \hat{T}\times\hat{W}$$
$$\vec{r}_{IJK} = \left[\hat{N} \,\vdots\, \hat{T} \,\vdots\, \hat{W}\right]\vec{r}_{NTW} \tag{8}$$

3. Transformation Details

To transform the covariance matrix between coordinate systems, a simple relationship exists using the transformation of the state between the coordinate systems. For conversions to different orbital format types, the process involves partial derivatives. I describe both in this section, but have placed the partial derivative equations in the appendix due to their length.

We begin by briefly describing the covariance matrix, and its characteristics. The covariance matrix is a product of the Least Squares process, using observational data taken on the satellite. This process is often illustrated by a straight line passing through certain observations, or random variables. When we consider these random variables as vectors, we have a matrix solution. Suppose we have a system in which the observations are $\boldsymbol{y}$, the state is $\boldsymbol{x}$, and we have arbitrary constants, m and $\boldsymbol{b}$. Using a linear relation,

$$\boldsymbol{y} = m\boldsymbol{x} + \boldsymbol{b} \tag{9}$$

Recall the definition of the *sample mean* as the expected value (E) of the random variable x

$$\bar{x} \equiv E(x) = \int_{-\infty}^{\infty} \xi p(\xi) d\xi \cong \frac{1}{N} \sum_{i=1}^{N} \xi_i \tag{10}$$

Using Eq. (10), we can determine the mean of the observations,

$$\bar{\boldsymbol{y}} \equiv E(\boldsymbol{y}) = E[m\boldsymbol{x} + b] = mE(\boldsymbol{x}) + \boldsymbol{b} = m\bar{\boldsymbol{x}} + \boldsymbol{b}$$

Recall that the ***sample variance***, σ^2, represents the variability of the expected value of each variable about the sample mean:

$$\begin{aligned} \sigma^2 &\equiv E\{[x - E(x)]^2\} = E\{[x - \bar{x}]^2\} \\ &= \int_{-\infty}^{\infty} (\xi - \bar{x}) p(\xi) d\xi \cong \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})^2 \end{aligned} \tag{11}$$

Using Eq. (11), we find the sample variance as follows. It's a square matrix due to the squared term in the expectation operator ($\boldsymbol{b}$ cancels when we substitute $\boldsymbol{y}$ and $\bar{\boldsymbol{y}}$):

$$\boldsymbol{P}_y \equiv E[(\boldsymbol{y} - \bar{\boldsymbol{y}})(\boldsymbol{y} - \bar{\boldsymbol{y}})^T] = E[m(\boldsymbol{x} - \bar{\boldsymbol{x}})(\boldsymbol{x} - \bar{\boldsymbol{x}})^T m^T]$$

Using properties from linear-error theory gives us

$$\boldsymbol{P}_y = mE[(\boldsymbol{x} - \bar{\boldsymbol{x}})(\boldsymbol{x} - \bar{\boldsymbol{x}})^T] m^T \tag{12}$$

This result is known as the ***similarity transformation***; it allows us mathematically to describe the covariance matrix, $\boldsymbol{P}_y$. In addition, m is known as the ***Jacobian*** for the transformation and the subscripts indicate the ending/initial states. This relation is essential for the transformation of covariance matrices.

$$\boldsymbol{P}_y = m\boldsymbol{P}_x m^T \qquad m_{\frac{y}{x}} = \frac{\partial y}{\partial x} \tag{13}$$

There are several important mathematical properties that we can use for analysis. First, the Jacobian matrix is orthogonal. This means that the inverse is equal to the transpose, and that its determinant is equal to unity. The covariance matrix is symmetric, implying that it can be diagonalized ($\boldsymbol{D} = \boldsymbol{B}^T\boldsymbol{P}\boldsymbol{B}$) where $\boldsymbol{D}$ is the diagonal matrix, and $\boldsymbol{B}$ is an orthogonal matrix related to the eigenvalues of $\boldsymbol{P}$.

In orbit determination, the covariance matrix results from the normal equations (Vallado, 2001, 693)

$$\hat{\boldsymbol{X}} = (\boldsymbol{A}^T\boldsymbol{W}\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{W}\boldsymbol{b} \tag{14}$$

where $\boldsymbol{A}$ is the partial derivative matrix, $\boldsymbol{b}$ is the residual matrix, and $\boldsymbol{W}$ is the weighting matrix. The matrix inverse, $(\boldsymbol{A}^T\boldsymbol{W}\boldsymbol{A})^{-1}$, is called the ***covariance matrix***, and I've shown the most general case by including the weighting matrix. Besides providing the best estimate of the state, the least-squares method also gives us statistical confidence in the uncertainty of the answers. The covariance matrix, $\boldsymbol{P} = (\boldsymbol{A}^T\boldsymbol{W}\boldsymbol{A})^{-1}$, contains the estimates of both variances and covariances for the closeness of the fit. The variances, the squares of the standard deviations, describe the closeness of the "fit". An example state having three estimated parameters—α, β, γ—gives us a covariance matrix of the form

$$\boldsymbol{P} \equiv (\boldsymbol{A}^T\boldsymbol{W}\boldsymbol{A})^{-1} = E([\boldsymbol{X} - \bar{\boldsymbol{X}}]^2) = \begin{bmatrix} \sigma_\alpha^2 & \mu_{\alpha\beta}\sigma_\alpha\sigma_\beta & \mu_{\alpha\gamma}\sigma_\alpha\sigma_\gamma \\ \mu_{\beta\alpha}\sigma_\alpha\sigma_\beta & \sigma_\beta^2 & \mu_{\beta\gamma}\sigma_\beta\sigma_\gamma \\ \mu_{\gamma\alpha}\sigma_\alpha\sigma_\gamma & \mu_{\gamma\beta}\sigma_\beta\sigma_\gamma & \sigma_\gamma^2 \end{bmatrix}$$

where σ_α is the standard deviation defined in Eq. (11), $\sigma_\alpha{}^2$ is the variance, $\mu_{\alpha\beta}$ is the correlation coefficient of α with β ($\mu_{\beta\alpha} = \mu_{\alpha\beta}$), and so forth.

The diagonal terms are the variances of the estimate of the state parameters. The square roots of the variances are the sample standard deviations for each element of the state space. You should include these values when discussing the results of an estimation calculation because they establish a level of uncertainty in each element. AFSPC didn't use this data for many decades because it was deemed inaccurate when obtained from sparse data with limited dynamics (SGP4). The covariance may indeed be inaccurate due to poorly known parameters (noise, biases, and drift) and inadequate propagation techniques used during differential correction, but these can be fixed. Whatever the fidelity, we can use the covariance matrix to indicate trends, such as which direction to search if a sensor doesn't immediately acquire a satellite. For orbit problems, the Jacobian assumes the form (where *oe* are any set of orbital elements)

$$J_{\frac{\hat{r},\hat{v}}{oe}} = \left[\frac{\partial}{\partial oe}\hat{r} \;\; \frac{\partial}{\partial oe}\hat{v}\right] \tag{15}$$

3.1 Orbit State Format Transformations

The process of taking the partial derivatives is laborious, but the modern computer is of some assistance. After the partial derivatives were formed, a check was conducted with MATHEMATICA. The equations are included in the appendix. Once these equations were programmed, it's relatively straightforward to convert the covariance matrices from one system to another. Matrix multiplications are used as in the preceding section, Eq. (13), although here the Jacobian is formed from the equations given in the appendix (and Eq. (15)) and the covariance is changed with Eq. (13).

Based on the selection of the equation for each partial derivative, certain restrictions exist for the subsequent transformations. Most notable are the eccentricity and inclination restrictions. The following guidelines are provided based on several runs to determine the accuracy, and where it degraded to the point that the routine could no longer be considered accurate.

Note that if the transformation from cartesian to equinoctial is made directly, the restrictions cited below (1, 2, 5, 6) are eliminated.

Cartesian to Classical and Classical to Cartesian, and Classical to Equinoctial:

1. If the eccentricity < 0.00001, the orbit is near circular and some of the partial derivatives will be poorly defined.
2. If the eccentricity < 0.0000001, the orbit is too near circular and the calculations should be stopped.
3. If the eccentricity > 0.9999, the orbit is near parabolic and some derivatives will be poorly defined.
4. If the eccentricity > 0.999999, the orbit is too near parabolic and the calculation should be stopped.
5. If the (inclination < 0.00001º) or (180º – inclination < 0.00001º), the orbit is near equatorial and some derivatives will be poorly defined.
6. If the (inclination < 0.00000001º) or (180º – inclination < 0.00000001º), the orbit is too near equatorial. the calculation will be stopped.

Equinoctial to Classical

These transformations require a different test to ensure accuracy.

1. if $|a_f| < 1 \times 10^{-6}$, if $|a_g| < 1 \times 10^{-6}$, the orbit is very circular. some derivatives are poorly defined and no covariance will be calculated.
2. if $|\chi| < 0.000001$, if $|\psi| < 0.000001$, the orbit is nearly equatorial. some derivatives are poorly defined and no covariance will be calculated.

Cartesian to Flight

1. if $(r_I^2+r_J^2) < 0.00001$, the orbit is directly over a pole and azimuth is undefined.

Flight to Cartesian

1. if sqrt($reci_I^2 + reci_J^2$) < 0.000001, then the satellite is directly over the pole and the longitude is undefined.

3.2 Coordinate System Transformations

For coordinate system transformations, we require just the transformation between the systems. For instance, if we wished to represent a cartesian covariance (ECI) in MOD, we would need to find the precession matrix that transforms an ECI state into the MOD state. After finding the precession angles (Θ, z, ζ), the complete rotation matrix for transformations from ECI to MOD is

$$[Prec] = \begin{bmatrix} \cos(\Theta)\cos(z)\cos(\zeta) - \sin(z)\sin(\zeta) & -\sin(\zeta)\cos(\Theta)\cos(z) - \sin(z)\cos(\zeta) & -\sin(\Theta)\cos(z) \\ \sin(z)\cos(\Theta)\cos(\zeta) + \sin(\zeta)\cos(z) & -\sin(z)\sin(\zeta)\cos(\Theta) + \cos(z)\cos(\zeta) & -\sin(\Theta)\sin(z) \\ \sin(\Theta)\cos(\zeta) & -\sin(\Theta)\sin(\zeta) & \cos(\Theta) \end{bmatrix}$$

Using matrix multiplications and a short hand notation where [Prec] and [0] are each 3x3 matrices, recognize this matrix as the Jacobian of the system.

$$[J] = \begin{bmatrix} Prec & 0 \\ 0 & Prec \end{bmatrix} \qquad \vec{P}_{MOD} = [J]\vec{P}_{ECI}[J]^T \tag{16}$$

Additional considerations are the transformations to the PEF and ECEF coordinate systems. These transformations include accounting for rotating coordinate systems and they require extra processing. Recall the transformation of the velocity vector from ECI to ECEF (Vallado, 2001, 221) with the shorthand for PM = polar motion, ST = sidereal time, NUT = nutation, PREC = precession, and $\omega_{\oplus}$ = Earth rotation rate.

$$\vec{r}_{PEF} = [ST][NUT][PREC]\vec{r}_{ECI}$$
$$\vec{r}_{ECEF} = [PM]\vec{r}_{PEF}$$
$$\vec{v}_{ECEF} = [PM]\left\{[ST][NUT][PREC]\vec{v}_{ECI} - \vec{\omega}_{\oplus} \times \vec{r}_{PEF}\right\}$$

An additional matrix is required to account for the rotation between the coordinate systems. If we cross the Earth rotation rate vector with the position vector (r_I, r_J, r_K) we obtain $-r_J\omega_{\oplus} + r_I\,\omega_{\oplus}$. Taking the "J" and "I" rows from the combined transformation matrix (sidereal time, nutation, and precession), and using $J_1(i,j)$ to represent the terms of this combined matrix,

$$J_1 = [ST][NUT][PREC]$$

$$[wr] = \omega_{\oplus}\begin{bmatrix} J_1(2,1) & J_1(2,2) & J_1(2,3) \\ -J_1(1,1) & -J_1(1,2) & -J_1(1,3) \\ 0 & 0 & 0 \end{bmatrix} \tag{17}$$

The transformation and conversion to PEF is found as before,

$$[J] = \begin{bmatrix} J_1 & 0 \\ wr & J_1 \end{bmatrix} \qquad \vec{P}_{PEF} = [J]\vec{P}_{ECI}[J]^T \tag{18}$$

This process is also used to convert to ECEF, but a final transformation is required using Eq. (13) and the polar motion transformation matrix.

When reversing the process (from ECEF), the first step is to convert to PEF using the polar motion matrix.

$$[J] = \begin{bmatrix} PM^T & 0 \\ 0 & PM^T \end{bmatrix} \qquad \vec{P}_{PEF} = [J]\vec{P}_{ECEF}[J]^T$$

The additional matrix for rotating coordinate systems is,

$$J_1 = [\text{PREC}]^T[\text{NUT}]^T[\text{ST}]^T$$

$$[wr] = \omega_{\oplus}\begin{bmatrix} J_1(1,2) & -J_1(1,1) & 0 \\ J_1(2,2) & -J_1(2,1) & 0 \\ J_1(3,2) & -J_1(3,1) & 0 \end{bmatrix} \qquad \vec{P}_{ECI} = [J_2]\vec{P}_{PEF}[J_2]^T \tag{19}$$

$$[J_2] = \begin{bmatrix} J_1 & 0 \\ wr & J_1 \end{bmatrix}$$

The satellite coordinate systems are simple transformations from the cartesian state. A variety of techniques can accomplish this transformation. I use the unit vector approach (Eq. (7) and Eq. (8)) and Eq. (13) in these formulations.

$$[J] = \begin{bmatrix} \left[\hat{R} \,\vdots\, \hat{S} \,\vdots\, \hat{W}\right]^T & 0 \\ 0 & \left[\hat{R} \,\vdots\, \hat{S} \,\vdots\, \hat{W}\right]^T \end{bmatrix} \qquad \vec{P}_{RSW} = \left[J\right]\vec{P}_{Cart}\left[J\right]^T$$

$$[J] = \begin{bmatrix} \left[\hat{N} \,\vdots\, \hat{T} \,\vdots\, \hat{W}\right]^T & 0 \\ 0 & \left[\hat{N} \,\vdots\, \hat{T} \,\vdots\, \hat{W}\right]^T \end{bmatrix} \qquad \vec{P}_{NTW} = \left[J\right]\vec{P}_{Cart}\left[J\right]^T \tag{20}$$

4. Analysis

The preceding equations were programmed in MATLAB, and many tests were run. The following checks were performed to test the accuracy of the transformations.

1. Basic Accuracy of the Orbit State Formats Conversions. Input a cartesian covariance. Convert to classical, equinoctial, and flight formats, and then convert back to cartesian. Check the accuracy of the results with initial starting cartesian covariance, and evaluate the transformation matrices and their inverses.
2. Basic Accuracy of the Coordinate System Conversions. Take an input cartesian covariance matrix and convert it to the alternate coordinate systems, and back. Evaluate the accuracy of the ending and the original covariance. Also examine the inverse of each transformation matrix with the transformation matrix calculated from the partial derivatives.
3. Satellite Results. Take representative satellite orbits and analyze the performance of the conversion from equinoctial to cartesian covariance, and back.
4. Covariance Propagation Results. Propagate the covariance using the full covariance (say for 8 days) and compare to propagating just the diagonal covariance using the position and velocity variances only.

Because several of the transformations use the gravitational parameter, I chose μ = 3.986004418 x 10^{14} m^3/s^2. Meters and meters per second are used as standard units throughout the analysis.

There are a variety of input and outputs used by organizations. Because the covariance matrix is [usually] symmetric, a common shorthand is to only provide the upper or lower triangular covariance matrix elements. It's quite important to understand which is in use for a particular problem as the elements of each do not line up, and the most common resulting error will be a “singular element on the diagonal” message. The “number” of each term is listed in the following matrices.

$$\begin{bmatrix} 1 & . & . & . & . & . \\ 2 & 3 & . & . & . & . \\ 4 & 5 & 6 & . & . & . \\ 7 & 8 & 9 & 10 & . & . \\ 11 & 12 & 13 & 14 & 15 & . \\ 16 & 17 & 18 & 19 & 20 & 21 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ . & 7 & 8 & 9 & 10 & 11 \\ . & . & 12 & 13 & 14 & 15 \\ . & . & . & 16 & 17 & 18 \\ . & . & . & . & 19 & 20 \\ . & . & . & . & . & 21 \end{bmatrix}$$

Some programs also output degrees for individual elements rather than radians. The practice is often undocumented, but will result in vastly different covariance matrices. Be sure to understand what units, if any, are implied within a certain program.

Finally, the order of the states is very important. I’ve listed certain orbital elements in a particular order, and the partial derivatives are in that order too. Changing the order of the state implies a commensurate change in the order of the covariance partial derivatives, as well as any orbital states that are required for a transformation. Classical orbital elements are generally consistent (a, e, i, Ω, ω, ν) with the only exception being the final element. The choice here is either the true anomaly (ν), mean anomaly (M), the eccentric anomaly (E), or time since periapsis ($\Delta\tau$). The equinoctial elements present a different case in that several “standards” exist. I’ve chosen to use the a_f, a_g, L, n, χ, ψ order. Flight parameters are usually ϕ_{gc}, λ, ϕ_{fpa}, β, r, and v.

4.1 Basic Accuracy Results - Orbit State Formats

The initial starting covariance was chosen as a simple diagonal matrix with 1.0 m errors in each position component and 0.001 m/s errors in each velocity component. In addition, the off-diagonal terms were set to non-zero values. Because the goal of this section was to test the accuracy of the transformations going to and from the cartesian form, zero values would have prevented evaluation of changes for the off-diagonal elements. The 3x3 sub-matrix values were set to include a sense of realism considering the initial accuracy of the diagonal terms. I used a percentage of the original cartesian covariance to evaluate the accuracy. Finally, I also evaluated the percentage difference between the inverse of the transformation matrix going from, and coming to the original covariance. This operation checked the accuracy of the partial derivatives. The initial covariance was:

```
cartesian covariance
       x  m            y m             z  m           xdot  m/s        ydot  m/s        zdot  m/s
   1.000000e+000   1.000000e-002   1.000000e-002   1.000000e-004   1.000000e-004   1.000000e-004
   1.000000e-002   1.000000e+000   1.000000e-002   1.000000e-004   1.000000e-004   1.000000e-004
   1.000000e-002   1.000000e-002   1.000000e+000   1.000000e-004   1.000000e-004   1.000000e-004
   1.000000e-004   1.000000e-004   1.000000e-004   1.000000e-006   1.000000e-006   1.000000e-006
   1.000000e-004   1.000000e-004   1.000000e-004   1.000000e-006   1.000000e-006   1.000000e-006
   1.000000e-004   1.000000e-004   1.000000e-004   1.000000e-006   1.000000e-006   1.000000e-006
```

The analysis requires a time and state vector (shown below). The following timing data (Earth Orientation Parameters, EOP, ΔUT1, ΔAT, x_p, y_p) was also used.

Consider the following state vector and equinoctial covariance.

$$\vec{r}_{IJK} = -605.79221660\,\hat{I} - 5870.22951108\,\hat{J} + 3493.05319896\,\hat{K} \text{ km}$$

$$\vec{v}_{IJK} = -1.56825429\,\hat{I} - 3.70234891\,\hat{J} - 6.47948395\,\hat{K} \text{ km/s}$$

Dec 15, 2000 16:58:50.208

ΔUT1 = 0.105970 s, ΔAT = 32 s, x_p = 0.000000 ", y_p = 0.000000 ", lod = 0.000000 s

From this information, the following orbital state formats were obtained using standard relations from Section 2.

	p km	a km	ecc	incl deg	raan deg	argp deg	nu deg	m deg
coes	6860.7554	6860.7631	0.0010640	97.65184	79.54701	83.86041	65.21303	65.10238

	af	ag	meanlon deg	n	chi	psi
eq	-0.0010197	0.0003038	228.5098015	0.0011110	1.1243593	0.2074336

	lon deg	latgc deg	fpa deg	az deg	magr km	magv km/s
flt	-75.2475117	30.6218751	0.0553210	-171.0988678	6857.6963605	7.6256489

Using a mean anomaly as the final element of the classical covariance, the conversion to classical covariance was as follows.

classical covariance

a m	ecc	incl rad	raan rad	argp rad	m rad
1.215911e+001	8.212505e-007	1.988270e-007	-1.526735e-007	1.159226e-003	-1.158147e-003
8.212505e-007	8.083254e-014	1.698441e-014	-1.304184e-014	7.796741e-011	-7.787492e-011
1.988270e-007	1.698441e-014	1.040397e-014	5.668433e-015	2.215181e-011	-2.212971e-011
-1.526735e-007	-1.304184e-014	5.668433e-015	1.859767e-014	-1.700668e-011	1.699277e-011
1.159226e-003	7.796741e-011	2.215181e-011	-1.700668e-011	1.202832e-007	-1.201791e-007
-1.158147e-003	-7.787492e-011	-2.212971e-011	1.699277e-011	-1.201791e-007	1.200752e-007

Converting this back to a cartesian format results in

cartesian covariance

x m	y m	z m	xdot m/s	ydot m/s	zdot m/s
1.000002e+000	1.000736e-002	1.000196e-002	1.000122e-004	1.000071e-004	1.000163e-004
1.000736e-002	1.000021e+000	1.001068e-002	1.000282e-004	1.000099e-004	1.000424e-004
1.000196e-002	1.001068e-002	9.999736e-001	1.000545e-004	1.000678e-004	1.000442e-004
1.000122e-004	1.000282e-004	1.000545e-004	9.999877e-007	9.999345e-007	1.000029e-006
1.000071e-004	1.000099e-004	1.000678e-004	9.999345e-007	9.998813e-007	9.999760e-007
1.000163e-004	1.000424e-004	1.000442e-004	1.000029e-006	9.999760e-007	1.000071e-006

Comparing to the original covariance, the following percentage differences were noted. In this analysis, individual differences, or values below 1×10^{-18} were considered to be negligible. This limit was determined by a study to evaluate the performance of the cartesian to equinoctial covariance transformation. The goal was to determine how 1 cm and 1 mm/s errors would translate to an equinoctial covariance. When these initial variances were input in the cartesian covariance, terms on the order of 1×10^{-16} to 1×10^{-21} were obtained in the equinoctial result, and hence the tolerance selection of 1×10^{-18}.

pct differences if over 1.000000e-018

-0.0002	-0.0736	-0.0196	-0.0122	-0.0071	-0.0163
-0.0736	-0.0021	-0.1068	-0.0282	-0.0099	-0.0424
-0.0196	-0.1068	0.0026	-0.0545	-0.0678	-0.0442
-0.0122	-0.0282	-0.0545	0.0012	0.0065	-0.0029
-0.0071	-0.0099	-0.0678	0.0065	0.0119	0.0024
-0.0163	-0.0424	-0.0442	-0.0029	0.0024	-0.0071

Although the percentages are all quite small, notice that the variations for most of the covariance terms are larger than those for the variances—a reason for the later analysis of the effect of these covariance terms on a covariance propagation.

The second test of accuracy was to look at the difference between the inverses of the transformation matrix. The transformation matrix going from cartesian to classical (ct2cl) was

tm ct2cl

-1.768332e-001	-1.713544e+000	1.019636e+000	-3.703851e+002	-8.744086e+002	-1.530303e+003
-3.261695e-008	-1.166025e-007	-8.128501e-008	-3.315927e-005	-1.553315e-004	-3.295410e-005
7.316107e-008	-1.349752e-008	-9.995027e-009	-1.096404e-004	2.022761e-005	1.497871e-005
1.229425e-007	-2.268173e-008	-1.679601e-008	6.627761e-005	-1.222759e-005	-9.054632e-006
8.613283e-007	-7.860480e-005	1.123284e-004	-4.143953e-002	-6.424291e-002	-2.166397e-001
-8.741531e-007	7.846095e-005	-1.123541e-004	4.141133e-002	6.418390e-002	2.164450e-001

and its inverse was

tm ct2cl inv

-8.829808e-002	2.551516e+005	3.435083e+006	5.870230e+006	-1.409737e+006	-1.411582e+006
-8.556234e-001	2.472462e+006	-6.337403e+005	-6.057922e+005	-3.323832e+006	-3.332477e+006
5.091348e-001	-1.471227e+006	-4.692898e+005	2.166082e-007	-5.830333e+006	-5.832170e+006
1.142915e-004	-1.269885e+003	-6.371951e+003	3.702349e+003	6.721174e+002	6.739326e+002
2.698205e-004	-7.476387e+003	1.175565e+003	-1.568254e+003	6.524030e+003	6.530521e+003
4.722131e-004	8.010540e+002	8.705152e+002	1.850021e-010	-3.890477e+003	-3.885957e+003

The transformation matrix going from classical to cartesian was

tm cl2ct

-8.829808e-002	2.551516e+005	3.435083e+006	5.870230e+006	-1.409737e+006	-1.411582e+006
-8.556234e-001	2.472462e+006	-6.337403e+005	-6.057922e+005	-3.323832e+006	-3.332476e+006
5.091348e-001	-1.471227e+006	-4.692898e+005	0.000000e+000	-5.830333e+006	-5.832170e+006
1.142915e-004	-1.269885e+003	-6.371951e+003	3.702349e+003	6.721174e+002	6.739326e+002
2.698205e-004	-7.476387e+003	1.175565e+003	-1.568254e+003	6.524030e+003	6.530521e+003
4.722131e-004	8.010540e+002	8.705152e+002	0.000000e+000	-3.890477e+003	-3.885957e+003

and the percentage difference between the inverse of the transformation going from cartesian to classical, and the transformation from classical to cartesian was as follows. Notice that the differences occur only where zero values were in the original matrix. A default of 100% was set in these cases.

pct differences if over 1.000000e-018

0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	100.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	100.0000	0.0000	0.0000

The flight (or spherical) transformation resulted in the following covariance matrices, and accuracy when converted back to cartesian.

flight covariance

lon rad	latgc rad	fpa rad	az rad	r m	v m/s
2.865506e-014	3.128972e-016	-4.983945e-015	1.330765e-014	-6.575239e-010	-2.329193e-011
3.128972e-016	2.147976e-014	-2.243944e-014	-1.639682e-015	-9.003905e-010	-3.189516e-011
-4.983945e-015	-2.243944e-014	2.734133e-014	4.342376e-015	3.449826e-009	1.222056e-010
1.330765e-014	-1.639682e-015	4.342376e-015	1.364980e-014	3.439029e-009	1.218231e-010
-6.575239e-010	-9.003905e-010	3.449826e-009	3.439029e-009	9.918921e-001	6.702467e-005
-2.329193e-011	-3.189516e-011	1.222056e-010	1.218231e-010	6.702467e-005	2.374262e-006

cartesian covariance

x m	y m	z m	xdot m/s	ydot m/s	zdot m/s
1.000000e+000	1.000000e-002	1.000000e-002	1.000000e-004	1.000000e-004	1.000000e-004
1.000000e-002	1.000000e+000	1.000000e-002	1.000000e-004	1.000000e-004	1.000000e-004
1.000000e-002	1.000000e-002	1.000000e+000	1.000000e-004	1.000000e-004	1.000000e-004
1.000000e-004	1.000000e-004	1.000000e-004	1.000000e-006	1.000000e-006	1.000000e-006
1.000000e-004	1.000000e-004	1.000000e-004	1.000000e-006	1.000000e-006	1.000000e-006
1.000000e-004	1.000000e-004	1.000000e-004	1.000000e-006	1.000000e-006	1.000000e-006

pct differences if over 1.000000e-018

0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

tm ct2fl

1.685560e-007	-1.739454e-008	0.000000e+000	0.000000e+000	0.000000e+000	0.000000e+000
7.624598e-009	7.388365e-008	1.254870e-007	0.000000e+000	0.000000e+000	0.000000e+000
-2.997654e-008	-7.067772e-008	-1.239758e-007	-1.155823e-005	-1.121921e-004	6.690357e-005
8.599334e-008	-8.885441e-009	-1.874737e-011	-1.278118e-004	2.358006e-005	1.746123e-005
-8.833757e-002	-8.560060e-001	5.093625e-001	0.000000e+000	0.000000e+000	0.000000e+000
0.000000e+000	0.000000e+000	0.000000e+000	-2.056552e-001	-4.855126e-001	-8.496961e-001

tm ct2fl inv

5.870230e+006	3.585696e+005	0.000000e+000	0.000000e+000	-8.833757e-002	0.000000e+000
-6.057922e+005	3.474600e+006	0.000000e+000	0.000000e+000	-8.560060e-001	0.000000e+000
3.446976e-011	5.901405e+006	0.000000e+000	0.000000e+000	5.093625e-001	0.000000e+000
3.702349e+003	-6.651333e+002	-6.721174e+002	-7.432315e+003	-1.994763e-019	-2.056552e-001
-1.568254e+003	-6.445255e+003	-6.524030e+003	1.371192e+003	-1.109613e-019	-4.855126e-001
-7.702471e-014	3.843775e+003	3.890477e+003	1.015379e+003	-1.157357e-019	-8.496961e-001

tm fl2ct

5.870230e+006	3.585696e+005	0.000000e+000	0.000000e+000	-8.833757e-002	0.000000e+000
-6.057922e+005	3.474600e+006	0.000000e+000	0.000000e+000	-8.560060e-001	0.000000e+000
0.000000e+000	5.901405e+006	0.000000e+000	0.000000e+000	5.093625e-001	0.000000e+000
3.702349e+003	-6.651333e+002	-6.721174e+002	-7.432315e+003	0.000000e+000	-2.056552e-001
-1.568254e+003	-6.445255e+003	-6.524030e+003	1.371192e+003	0.000000e+000	-4.855126e-001
0.000000e+000	3.843775e+003	3.890477e+003	1.015379e+003	0.000000e+000	-8.496961e-001

-------- tm accuracy ---------

pct differences if over 1.000000e-018

0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
100.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
100.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Although not used as often, this transformation proved to be the best in terms of accuracy.

Using the classical covariance obtained earlier, the equinoctial covariance results were as follows.

equinoctial covariance

af	ag	meanlon rad	n	chi	psi
1.307401e-013	8.454837e-014	-1.025769e-013	2.767180e-016	-2.243993e-014	-2.467241e-014
8.454837e-014	8.622244e-014	-6.605814e-014	2.301159e-016	-1.730851e-014	-1.900285e-014
-1.025769e-013	-6.605814e-014	1.021784e-013	-2.251061e-016	3.247419e-014	5.462808e-016
2.767180e-016	2.301159e-016	-2.251061e-016	7.173977e-019	-4.709651e-017	-5.180442e-017
-2.243993e-014	-1.730851e-014	3.247419e-014	-4.709651e-017	1.685802e-014	-8.851445e-015
-2.467241e-014	-1.900285e-014	5.462808e-016	-5.180442e-017	-8.851445e-015	2.129877e-014

```
classical covariance
          a  m            ecc            incl rad      raan rad          argp rad          m rad
   1.215911e+001   8.212505e-007   1.988270e-007  -1.526735e-007   1.159226e-003  -1.158147e-003
   8.212505e-007   8.083254e-014   1.698441e-014  -1.304184e-014   7.796741e-011  -7.787492e-011
   1.988270e-007   1.698441e-014   1.040397e-014   5.668433e-015   2.215181e-011  -2.212971e-011
  -1.526735e-007  -1.304184e-014   5.668433e-015   1.859767e-014  -1.700668e-011   1.699277e-011
   1.159226e-003   7.796741e-011   2.215181e-011  -1.700668e-011   1.202832e-007  -1.201791e-007
  -1.158147e-003  -7.787492e-011  -2.212971e-011   1.699277e-011  -1.201791e-007   1.200752e-007
-------- accuracy ---------
pct differences if over 1.000000e-018
        0.0000        0.0000        0.0000        0.0000        0.0000        0.0000
        0.0000        0.0000        0.0000        0.0000        0.0000        0.0000
        0.0000        0.0000        0.0000        0.0000        0.0000        0.0000
        0.0000        0.0000        0.0000        0.0000        0.0000        0.0000
        0.0000        0.0000        0.0000        0.0000        0.0000        0.0000
        0.0000        0.0000        0.0000        0.0000        0.0000        0.0000
tm cl2eq
   0.000000e+000  -9.583596e-001   0.000000e+000  -3.038363e-004  -3.038363e-004   0.000000e+000
   0.000000e+000   2.855642e-001   0.000000e+000  -1.019681e-003  -1.019681e-003   0.000000e+000
   0.000000e+000   0.000000e+000   0.000000e+000   1.000000e+000   1.000000e+000   1.000000e+000
  -2.429009e-010   0.000000e+000   0.000000e+000   0.000000e+000   0.000000e+000   0.000000e+000
   0.000000e+000   0.000000e+000   1.134461e+000   2.074336e-001   0.000000e+000   0.000000e+000
   0.000000e+000   0.000000e+000   2.092973e-001  -1.124359e+000   0.000000e+000   0.000000e+000
tm cl2eq inv
   0.000000e+000   0.000000e+000   0.000000e+000  -4.116906e+009   0.000000e+000   0.000000e+000
  -9.583596e-001   2.855642e-001   3.419896e-020   0.000000e+000  -2.798202e-021   1.516718e-020
   0.000000e+000   0.000000e+000   0.000000e+000   0.000000e+000   8.524607e-001   1.572709e-001
   0.000000e+000   0.000000e+000   0.000000e+000   0.000000e+000   1.586839e-001  -8.601197e-001
  -2.683910e+002  -9.007260e+002  -2.081668e-017   0.000000e+000  -1.586839e-001   8.601197e-001
   2.683910e+002   9.007260e+002   1.000000e+000   0.000000e+000   8.221029e-018  -4.456071e-017
tm eq2cl
   0.000000e+000   0.000000e+000   0.000000e+000  -4.116906e+009   0.000000e+000   0.000000e+000
  -9.583596e-001   2.855642e-001   0.000000e+000   0.000000e+000   0.000000e+000   0.000000e+000
   0.000000e+000   0.000000e+000   0.000000e+000   0.000000e+000   8.524607e-001   1.572709e-001
   0.000000e+000   0.000000e+000   0.000000e+000   0.000000e+000   1.586839e-001  -8.601197e-001
  -2.683910e+002  -9.007260e+002   0.000000e+000   0.000000e+000  -1.586839e-001   8.601197e-001
   2.683910e+002   9.007260e+002   1.000000e+000   0.000000e+000   0.000000e+000   0.000000e+000
-------- accuracy tm ---------
pct differences if over 1.000000e-018
        0.0000        0.0000        0.0000        0.0000        0.0000        0.0000
        0.0000        0.0000        0.0000        0.0000        0.0000        0.0000
        0.0000        0.0000        0.0000        0.0000        0.0000        0.0000
        0.0000        0.0000        0.0000        0.0000        0.0000        0.0000
        0.0000        0.0000      100.0000        0.0000        0.0000        0.0000
        0.0000        0.0000        0.0000        0.0000      100.0000      100.0000
```

4.2 Basic Accuracy - Coordinate System Results

In this step, I calculated the covariance in different coordinate systems. For consistency, I used the same initial state and cartesian covariance data from Section 4.1, but I changed the EOP data slightly to permit an evaluation of the ECEF and PEF transformations.

ΔUT1 = 0.1032220 s, ΔAT = 32 s, x_p = -0.080171 ", y_p = 0.361253 ", lod = 0.000745 s

Notice that because each of these transformations results from the multiplication of orthogonal matrices (and not partial derivatives as in the previous section), the inverse and the transpose of the transformations will be equal ($\boldsymbol{R}^T = \boldsymbol{R}^{-1}$). This property was checked with each step and found to be true. Each coordinate system contains information of the state in that coordinate system, and the resulting covariance. The percentage differences were calculated as before, and all were 0.0 (above the $1\text{x}10^{-18}$ threshold). Note that for the PEF, ECEF and TEME coordinate systems, the kinematic terms for the apparent sidereal time were not used (Vallado, 2001, 219).

```
cartesian covariance in MOD
               x m           y m            z m            xdot  m/s     ydot  m/s     zdot  m/s
eci-mod     -604.8616829 -5870.3589279  3492.9969618 v   -1.566860729  -3.702684048  -6.479629582
cartesian covariance in MOD
        x  m            y m             z  m            xdot  m/s       ydot  m/s       zdot  m/s
   9.999939e-001   9.999070e-003   9.997861e-003   9.993866e-005   9.999070e-005   9.997861e-005
   9.999070e-003   1.000004e+000   1.000307e-002   9.999070e-005   1.000428e-004   1.000307e-004
   9.997861e-003   1.000307e-002   1.000002e+000   9.997861e-005   1.000307e-004   1.000186e-004
   9.993866e-005   9.999070e-005   9.997861e-005   9.993866e-007   9.999070e-007   9.997861e-007
   9.999070e-005   1.000428e-004   1.000307e-004   9.999070e-007   1.000428e-006   1.000307e-006
   9.997861e-005   1.000307e-004   1.000186e-004   9.997861e-007   1.000307e-006   1.000186e-006
```

```
----- transformation matrix
   1.000000e+000  -2.137966e-004  -9.290377e-005   0.000000e+000   0.000000e+000   0.000000e+000
   2.137966e-004   1.000000e+000  -9.931272e-009   0.000000e+000   0.000000e+000   0.000000e+000
   9.290377e-005  -9.931239e-009   1.000000e+000   0.000000e+000   0.000000e+000   0.000000e+000
   0.000000e+000   0.000000e+000   0.000000e+000   1.000000e+000  -2.137966e-004  -9.290377e-005
   0.000000e+000   0.000000e+000   0.000000e+000   2.137966e-004   1.000000e+000  -9.931272e-009
   0.000000e+000   0.000000e+000   0.000000e+000   9.290377e-005  -9.931239e-009   1.000000e+000
```

cartesian covariance in TOD

```
               x m            y m            z m            xdot  m/s     ydot  m/s     zdot  m/s
eci-tod     -605.1838381 -5870.2615478  3493.1048160 v   -1.567342331  -3.702665784  -6.479523542
cartesian covariance in TOD
        x  m            y m             z  m            xdot  m/s       ydot  m/s       zdot  m/s
   9.999960e-001   9.999542e-003   9.998451e-003   9.995987e-005   9.999542e-005   9.998451e-005
   9.999542e-003   1.000003e+000   1.000201e-002   9.999542e-005   1.000310e-004   1.000201e-004
   9.998451e-003   1.000201e-002   1.000001e+000   9.998451e-005   1.000201e-004   1.000092e-004
   9.995987e-005   9.999542e-005   9.998451e-005   9.995987e-007   9.999542e-007   9.998451e-007
   9.999542e-005   1.000310e-004   1.000201e-004   9.999542e-007   1.000310e-006   1.000201e-006
   9.998451e-005   1.000201e-004   1.000092e-004   9.998451e-007   1.000201e-006   1.000092e-006
----- transformation matrix
   1.000000e+000  -1.398390e-004  -6.083941e-005   0.000000e+000   0.000000e+000   0.000000e+000
   1.398399e-004   1.000000e+000   1.506393e-005   0.000000e+000   0.000000e+000   0.000000e+000
   6.083730e-005  -1.507244e-005   1.000000e+000   0.000000e+000   0.000000e+000   0.000000e+000
   0.000000e+000   0.000000e+000   0.000000e+000   1.000000e+000  -1.398390e-004  -6.083941e-005
   0.000000e+000   0.000000e+000   0.000000e+000   1.398399e-004   1.000000e+000   1.506393e-005
   0.000000e+000   0.000000e+000   0.000000e+000   6.083730e-005  -1.507244e-005   1.000000e+000
```

cartesian covariance in PEF

```
without 2 extra ast terms
               x m            y m            z m            xdot  m/s     ydot  m/s     zdot  m/s
eci-pef     1502.7504376 -5706.8344325  3493.1048160 v   -0.577822427  -4.127063788  -6.479523542
cartesian covariance in PEF
        x  m            y m             z  m            xdot  m/s       ydot  m/s       zdot  m/s
   9.934002e-001   7.512598e-003   5.831364e-003   3.400170e-005   7.512591e-005   5.831364e-005
   7.512598e-003   1.006599e+000   1.288427e-002   7.512605e-005   1.659892e-004   1.288427e-004
   5.831364e-003   1.288427e-002   1.000001e+000   5.831364e-005   1.288427e-004   1.000092e-004
   3.400170e-005   7.512605e-005   5.831364e-005   3.400170e-007   7.512598e-007   5.831364e-007
   7.512591e-005   1.659892e-004   1.288427e-004   7.512598e-007   1.659891e-006   1.288427e-006
   5.831364e-005   1.288427e-004   1.000092e-004   5.831364e-007   1.288427e-006   1.000092e-006
----- transformation matrix
   9.357735e-001  -3.526016e-001  -6.224450e-005   0.000000e+000   0.000000e+000   0.000000e+000
   3.526016e-001   9.357735e-001  -7.346940e-006   0.000000e+000   0.000000e+000   0.000000e+000
   6.083730e-005  -1.507244e-005   1.000000e+000   0.000000e+000   0.000000e+000   0.000000e+000
   2.571211e-011   6.823768e-011  -5.357473e-016   9.357735e-001  -3.526016e-001  -6.224450e-005
  -6.823768e-011   2.571211e-011   4.538940e-015   3.526016e-001   9.357735e-001  -7.346940e-006
   0.000000e+000   0.000000e+000   0.000000e+000   6.083730e-005  -1.507244e-005   1.000000e+000
```

cartesian covariance in ECEF

```
without 2 extra ast terms
               x m            y m            z m            xdot  m/s     ydot  m/s     zdot  m/s
eci-ecef    1502.7490799 -5706.8405503  3493.0954051 v   -0.577819908  -4.127052440  -6.479530995
cartesian covariance in ECEF
        x  m            y m             z  m            xdot  m/s       ydot  m/s       zdot  m/s
   9.934002e-001   7.512583e-003   5.831375e-003   3.400165e-005   7.512575e-005   5.831375e-005
   7.512583e-003   1.006599e+000   1.288428e-002   7.512590e-005   1.659887e-004   1.288428e-004
   5.831375e-003   1.288428e-002   1.000001e+000   5.831375e-005   1.288428e-004   1.000096e-004
   3.400165e-005   7.512590e-005   5.831375e-005   3.400165e-007   7.512583e-007   5.831375e-007
   7.512575e-005   1.659887e-004   1.288428e-004   7.512583e-007   1.659887e-006   1.288428e-006
   5.831375e-005   1.288428e-004   1.000096e-004   5.831375e-007   1.288428e-006   1.000096e-006
----- transformation matrix
   1.000000e+000  -6.807357e-013  -3.886800e-007   0.000000e+000   0.000000e+000   0.000000e+000
   0.000000e+000   1.000000e+000  -1.751404e-006   0.000000e+000   0.000000e+000   0.000000e+000
   3.886800e-007   1.751404e-006   1.000000e+000   0.000000e+000   0.000000e+000   0.000000e+000
   0.000000e+000   0.000000e+000   0.000000e+000   1.000000e+000  -6.807357e-013  -3.886800e-007
   0.000000e+000   0.000000e+000   0.000000e+000   0.000000e+000   1.000000e+000  -1.751404e-006
   0.000000e+000   0.000000e+000   0.000000e+000   3.886800e-007   1.751404e-006   1.000000e+000
```

cartesian covariance in TEME

```
order   4 terms   0 nutation option  a
               x m            y m            z m            xdot  m/s     ydot  m/s     zdot  m/s
eci-teme    -604.7497358 -5870.3065145  3493.1044297 v   -1.567068394  -3.702781274  -6.479523803
cartesian covariance in TEME
        x  m            y m             z  m            xdot  m/s       ydot  m/s       zdot  m/s
   1.000001e+000   1.000047e-002   9.999850e-003   1.000064e-004   1.000047e-004   9.999850e-005
   1.000047e-002   1.000000e+000   9.999679e-003   1.000047e-004   1.000030e-004   9.999679e-005
   9.999850e-003   9.999679e-003   9.999991e-001   9.999850e-005   9.999679e-005   9.999059e-005
   1.000064e-004   1.000047e-004   9.999850e-005   1.000064e-006   1.000047e-006   9.999850e-007
   1.000047e-004   1.000030e-004   9.999679e-005   1.000047e-006   1.000030e-006   9.999679e-007
   9.999850e-005   9.999679e-005   9.999059e-005   9.999850e-007   9.999679e-007   9.999059e-007
```

```
----- transformation matrix
   1.000000e+000  -4.333859e-015   3.204892e-005   0.000000e+000   0.000000e+000   0.000000e+000
  -4.808905e-010   1.000000e+000   1.500502e-005   0.000000e+000   0.000000e+000   0.000000e+000
  -3.204892e-005  -1.500502e-005   1.000000e+000   0.000000e+000   0.000000e+000   0.000000e+000
   0.000000e+000   0.000000e+000   0.000000e+000   1.000000e+000  -4.333859e-015   3.204892e-005
   0.000000e+000   0.000000e+000   0.000000e+000  -4.808905e-010   1.000000e+000   1.500502e-005
   0.000000e+000   0.000000e+000   0.000000e+000  -3.204892e-005  -1.500502e-005   1.000000e+000
```

cartesian covariance in RSW

```
               x m           y m            z m            xdot   m/s     ydot   m/s     zdot   m/s
eci-rsw    6857.6963605     0.0000000     0.0000000 v    0.007362813    7.625645351    0.000000000
cartesian covariance in RSW
        x  m            y m              z  m            xdot  m/s        ydot  m/s        zdot  m/s
   9.918921e-001   6.700644e-003  -2.878187e-003   1.892086e-005   6.700644e-005  -2.878187e-005
   6.700644e-003   1.013730e+000  -1.019283e-002   6.700644e-005   2.372970e-004  -1.019283e-004
  -2.878187e-003  -1.019283e-002   9.943782e-001  -2.878187e-005  -1.019283e-004   4.378217e-005
   1.892086e-005   6.700644e-005  -2.878187e-005   1.892086e-007   6.700644e-007  -2.878187e-007
   6.700644e-005   2.372970e-004  -1.019283e-004   6.700644e-007   2.372970e-006  -1.019283e-006
  -2.878187e-005  -1.019283e-004   4.378217e-005  -2.878187e-007  -1.019283e-006   4.378217e-007
----- transformation matrix
  -8.833757e-002  -8.560060e-001   5.093625e-001   0.000000e+000   0.000000e+000   0.000000e+000
  -2.055700e-001  -4.846864e-001  -8.501883e-001   0.000000e+000   0.000000e+000   0.000000e+000
   9.746473e-001  -1.798132e-001  -1.331531e-001   0.000000e+000   0.000000e+000   0.000000e+000
   0.000000e+000   0.000000e+000   0.000000e+000  -8.833757e-002  -8.560060e-001   5.093625e-001
   0.000000e+000   0.000000e+000   0.000000e+000  -2.055700e-001  -4.846864e-001  -8.501883e-001
   0.000000e+000   0.000000e+000   0.000000e+000   9.746473e-001  -1.798132e-001  -1.331531e-001
```

cartesian covariance in NTW

```
               x m           y m            z m            xdot   m/s     ydot   m/s     zdot   m/s
eci-ntw    6857.6931640     6.6213296     0.0000000 v    0.000000000    7.625648905    0.000000000
cartesian covariance in NTW
        x  m            y m              z  m            xdot  m/s        ydot  m/s        zdot  m/s
   9.918792e-001   6.679546e-003  -2.868345e-003   1.879167e-005   6.679546e-005  -2.868345e-005
   6.679546e-003   1.013743e+000  -1.019560e-002   6.679546e-005   2.374262e-004  -1.019560e-004
  -2.868345e-003  -1.019560e-002   9.943782e-001  -2.868345e-005  -1.019560e-004   4.378217e-005
   1.879167e-005   6.679546e-005  -2.868345e-005   1.879167e-007   6.679546e-007  -2.868345e-007
   6.679546e-005   2.374262e-004  -1.019560e-004   6.679546e-007   2.374262e-006  -1.019560e-006
  -2.868345e-005  -1.019560e-004   4.378217e-005  -2.868345e-007  -1.019560e-006   4.378217e-007
----- transformation matrix
  -8.813904e-002  -8.555377e-001   5.101831e-001   0.000000e+000   0.000000e+000   0.000000e+000
  -2.056552e-001  -4.855126e-001  -8.496961e-001   0.000000e+000   0.000000e+000   0.000000e+000
   9.746473e-001  -1.798132e-001  -1.331531e-001   0.000000e+000   0.000000e+000   0.000000e+000
   0.000000e+000   0.000000e+000   0.000000e+000  -8.813904e-002  -8.555377e-001   5.101831e-001
   0.000000e+000   0.000000e+000   0.000000e+000  -2.056552e-001  -4.855126e-001  -8.496961e-001
   0.000000e+000   0.000000e+000   0.000000e+000   9.746473e-001  -1.798132e-001  -1.331531e-001
```

For the RSW and NTW cases, the initial covariance was not changed. Because the original covariance was spherical, little change is noted in the transformations.

4.3 Satellite Results

The next test was to examine how the transformations worked for a variety of satellite orbits. A few cases were chosen—some good and some stressing. The determination of performance was based on taking a starting covariance in equinoctial elements, converting it to a cartesian format, and then converting back to equinoctial elements. The tolerance on the accuracy was again set at $1\text{x}10^{-18}$. The NTW covariance provided estimates on the accuracy of the initial equinoctial state.

```
Case 1:
year  2000 mon   12 day  15 hr  16:58:50.208000
dut1 0.103222 s dat  32 s xp -0.080171 " yp 0.361253 " lod 0.000745 s
r      -605.7922166 -5870.2295111  3493.0531990 v   -1.568254290  -3.702348910  -6.479483950
          p km       a km      ecc      incl deg     raan deg     argp deg      nu deg      m deg
coes   6860.7554  6860.7631  0.0010640   97.65184   79.54701   79.54701   83.86041   65.21303
           af            ag           meanlon           n            chi         psi
eq       -0.0010197      0.0003038   228.5098015      0.0011110      1.1243593     0.2074336
       lon deg       latgc deg         fpa deg        az deg       magr km      magv km/s
flt     -75.2475274     30.6217837     0.0553210   -171.0988678  6857.6963605     7.6256489
input equinoctial covariance
         af             ag          meanlon rad          n              chi              psi
   8.042040e-013   7.419230e-013  -2.026230e-012   3.787930e-016  -1.773020e-013   2.483520e-013
   7.419230e-013   2.190440e-012  -3.820340e-012   1.190100e-015  -2.838440e-013   3.679250e-013
  -2.026230e-012  -3.820340e-012   1.975700e-011   9.354380e-015   1.473860e-012  -3.015420e-012
   3.787930e-016   1.190100e-015   9.354380e-015   1.562360e-017  -8.860930e-017  -4.569740e-016
  -1.773020e-013  -2.838440e-013   1.473860e-012  -8.860930e-017   9.677970e-013  -7.230720e-013
   2.483520e-013   3.679250e-013  -3.015420e-012  -4.569740e-016  -7.230720e-013   1.841230e-012
```

```
cartesian covariance
        x  m            y m           z  m         xdot  m/s       ydot  m/s       zdot  m/s
   3.084175e+002  -3.631010e+002  -2.921318e+001   4.186625e-001  -2.143152e-001  -1.198201e-002
  -3.631010e+002   5.091000e+002   3.133073e+001  -5.787611e-001   2.476548e-001   1.107400e-002
  -2.921318e+001   3.133073e+001   7.692587e+001  -3.834765e-002   2.864365e-002   1.352535e-002
   4.186625e-001  -5.787611e-001  -3.834765e-002   6.967966e-004  -3.009157e-004  -1.212316e-005
  -2.143152e-001   2.476548e-001   2.864365e-002  -3.009157e-004   1.767174e-004   7.756206e-006
  -1.198201e-002   1.107400e-002   1.352535e-002  -1.212316e-005   7.756206e-006   2.765174e-005
equinoctial covariance
        af              ag         meanlon rad          n              chi            psi
   8.042040e-013   7.419230e-013  -2.026255e-012   3.787930e-016  -1.773020e-013   2.483520e-013
   7.419230e-013   2.190440e-012  -3.820402e-012   1.190100e-015  -2.838440e-013   3.679250e-013
  -2.026255e-012  -3.820402e-012   1.975723e-011   9.354346e-015   1.473869e-012  -3.015431e-012
   3.787930e-016   1.190100e-015   9.354346e-015   1.562360e-017  -8.860930e-017  -4.569740e-016
  -1.773020e-013  -2.838440e-013   1.473869e-012  -8.860930e-017   9.677970e-013  -7.230720e-013
   2.483520e-013   3.679250e-013  -3.015431e-012  -4.569740e-016  -7.230720e-013   1.841230e-012
-------- accuracy ---------
pct differences if over 1.000000e-018
        0.0000         0.0000       -0.0012        0.0000         0.0000         0.0000
        0.0000         0.0000       -0.0016        0.0000         0.0000         0.0000
       -0.0012        -0.0016       -0.0011        0.0000        -0.0006        -0.0004
        0.0000         0.0000        0.0000        0.0000         0.0000         0.0000
        0.0000         0.0000       -0.0006        0.0000         0.0000         0.0000
        0.0000         0.0000       -0.0004        0.0000         0.0000         0.0000
ntw      17.582        22.548        8.771  magnitude = 29.907 m
```

Case 2: This case was a more eccentric orbit (Molnyia).

```
year  2000 mon   12 day  15 hr   4:59:9.164000
dut1 0.106000 s dat  32 s xp 0.000000 " yp 0.000000 " lod 0.000000 s
r     16091.9939260 -5269.8969797 28254.8221721 v    0.257738430   1.895011970  -2.218700840
          p km       a km      ecc       incl deg     raan deg     argp deg      nu deg      m deg
coes  11575.1639 25516.4703  0.7391651   62.03466  224.23366  224.23366  255.13311  208.66039
         af           ag          meanlon          n           chi        psi
eq       -0.3624854     0.6441811     45.1201104      0.0001549    -0.4194393    -0.4308122
      lon deg       latgc deg        fpa deg       az deg       magr km      magv km/s
flt    -177.0731848    59.0695940   -45.2530046    114.1831450 32940.2346481     2.9291864
input equinoctial covariance
        af              ag         meanlon rad          n              chi            psi
   3.069620e-010   1.118810e-010  -5.309700e-010  -9.864510e-014   2.133450e-010  -2.026760e-010
   1.118810e-010   4.380630e-011  -2.250290e-010  -8.208040e-014   4.787440e-011  -1.027690e-010
  -5.309700e-010  -2.250290e-010   2.571060e-009   2.662620e-012  -3.981020e-010   1.182000e-010
  -9.864510e-014  -8.208040e-014   2.662620e-012   4.029670e-015  -1.260980e-013  -3.862790e-013
   2.133450e-010   4.787440e-011  -3.981020e-010  -1.260980e-013   1.001510e-009   7.458330e-010
  -2.026760e-010  -1.027690e-010   1.182000e-010  -3.862790e-013   7.458330e-010   1.208400e-009
cartesian covariance
        x  m            y m           z  m         xdot  m/s       ydot  m/s       zdot  m/s
   9.061815e+007  -6.640345e+006  -2.323751e+006   3.678240e+003  -2.582722e+003   2.596870e+002
  -6.640345e+006   6.405947e+005   2.454092e+005  -2.856776e+002   1.772842e+002  -1.082125e+001
  -2.323751e+006   2.454092e+005   4.049444e+006  -1.119958e+002   5.525219e+001  -3.173814e+002
   3.678240e+003  -2.856776e+002  -1.119958e+002   1.512055e-001  -1.035552e-001   9.999482e-003
  -2.582722e+003   1.772842e+002   5.525219e+001  -1.035552e-001   7.459182e-002  -7.771100e-003
   2.596870e+002  -1.082125e+001  -3.173814e+002   9.999482e-003  -7.771100e-003   2.834448e-002
equinoctial covariance
        af              ag         meanlon rad          n              chi            psi
   6.833355e-010   4.513370e-010  -1.306110e-009  -4.824364e-013   4.639489e-010  -1.970763e-010
   4.513370e-010   3.066603e-010  -8.593925e-010  -2.980426e-013   2.742721e-010  -1.827452e-010
  -1.306110e-009  -8.593925e-010   4.070660e-009   3.258402e-012  -9.147807e-010   2.337891e-010
  -4.824364e-013  -2.980426e-013   3.258402e-012   4.029670e-015  -3.827680e-013  -1.363848e-013
   4.639489e-010   2.742721e-010  -9.147807e-010  -3.827680e-013   1.168368e-009   7.502976e-010
  -1.970763e-010  -1.827452e-010   2.337891e-010  -1.363848e-013   7.502976e-010   1.041542e-009
-------- accuracy ---------
pct differences if over 1.000000e-018
     -122.6124     -303.4081     -145.9857     -389.0628     -117.4641        2.7629
     -303.4081     -600.0371     -281.9030     -263.1105     -472.8994      -77.8213
     -145.9857     -281.9030      -58.3262      -22.3758     -129.7855      -97.7911
     -389.0628     -263.1105      -22.3758        0.0000     -203.5481       64.6927
     -117.4641     -472.8994     -129.7855     -203.5481      -16.6607       -0.5986
        2.7629      -77.8213      -97.7911       64.6927       -0.5986       13.8082
ntw      31.980       108.976       17.301  magnitude = 114.881 m
```

This particular case presents some interesting questions concerning the transformations. Admittedly, the values are small by themselves, but remember that the covariance matrices are the product of a differential correction process. Because covariance matrices come from imperfect observations and a differential correction process that tries to find the ‘best” fit, it’s possible that the resulting covariance matrix can be inconsistent when processed through a series of transformations.

Case 3: This orbit was a retrograde orbit.

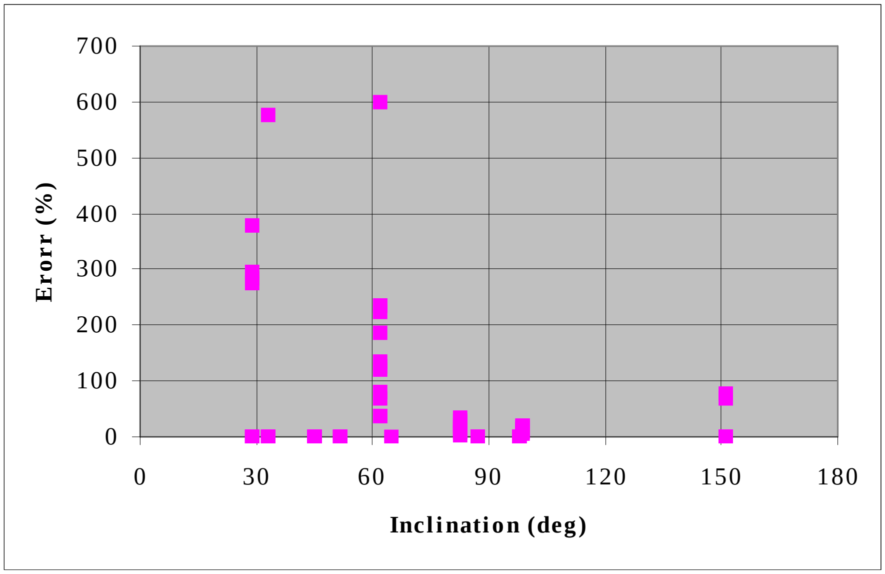
```
year  2000 mon   12 day  14 hr   5:25:3.461000
dut1 0.105970 s dat  32 s xp 0.000000 " yp 0.000000 " lod 0.000000 s
r      4364.5152493  4748.1760294  2430.2042765 v    5.879624140  -4.102949440  -2.535278190
         p km       a km      ecc      incl deg     raan deg     argp deg      nu deg      m deg
coes   6891.6486  6891.6495  0.0003655  150.96470  184.66064  184.66064   34.72333   98.68199
          af           ag            meanlon           n           chi        psi
eq       -0.0002825    -0.0002319   318.0245529      0.0011035    -0.3137868    -3.8490362
      lon deg       latgc deg         fpa deg       az deg       magr km      magv km/s
flt    -117.0362246    20.6487723     0.0207039   -110.8793408  6892.0288589     7.6047229
input equinoctial covariance
        af             ag          meanlon rad          n              chi            psi
   4.689140e-011   1.600900e-011   1.647310e-010  -4.381410e-016  -1.411950e-010   1.099990e-011
   1.600900e-011   1.068810e-011   6.397320e-011   6.531240e-016  -5.605540e-011   2.560990e-011
   1.647310e-010   6.397320e-011   6.713360e-010   1.044550e-014  -6.375070e-010   9.405540e-011
  -4.381410e-016   6.531240e-016   1.044550e-014   2.108970e-017   1.592720e-014   1.030810e-014
  -1.411950e-010  -5.605540e-011  -6.375070e-010   1.592720e-014   7.598440e-010  -1.054370e-010
   1.099990e-011   2.560990e-011   9.405540e-011   1.030810e-014  -1.054370e-010   1.864720e-010
cartesian covariance
       x  m            y m              z  m           xdot  m/s       ydot  m/s       zdot  m/s
   1.020635e+003  -2.913050e+003  -2.668700e+002   4.248847e+000  -9.381776e-001  -2.402350e-001
  -2.913050e+003   1.187780e+004   1.114968e+003  -1.807213e+001   2.943803e+000   1.387416e+000
  -2.668700e+002   1.114968e+003   2.993203e+002  -1.780953e+000   3.324118e-001   2.454211e-001
   4.248847e+000  -1.807213e+001  -1.780953e+000   2.789553e-002  -4.389358e-003  -2.334968e-003
  -9.381776e-001   2.943803e+000   3.324118e-001  -4.389358e-003   9.388165e-004   2.724837e-004
  -2.402350e-001   1.387416e+000   2.454211e-001  -2.334968e-003   2.724837e-004   5.001989e-004
equinoctial covariance
        af             ag          meanlon rad          n              chi            psi
   4.685739e-011   1.602275e-011   9.135764e-011  -4.363349e-016   1.410191e-010  -1.201412e-011
   1.602275e-011   1.070507e-011   3.399642e-011   6.509239e-016   5.957670e-011   1.619163e-011
   9.135764e-011   3.399642e-011   2.126508e-010   1.823300e-014   2.428233e-010  -7.647685e-012
  -4.363349e-016   6.509239e-016   1.823300e-014   2.108970e-017  -1.404728e-014   1.275172e-014
   1.410191e-010   5.957670e-011   2.428233e-010  -1.404728e-014   7.785063e-010   8.261788e-012
  -1.201412e-011   1.619163e-011  -7.647685e-012   1.275172e-014   8.261788e-012   1.678097e-010
-------- accuracy ---------
pct differences if over 1.000000e-018
        0.0725       -0.0859       44.5413        0.4122      199.8755      209.2203
       -0.0859       -0.1588       46.8583        0.3369      206.2818       36.7759
       44.5413       46.8583       68.3242      -74.5536      138.0895      108.1310
        0.4122        0.3369      -74.5536        0.0000      188.1968      -23.7059
      199.8755      206.2818      138.0895      188.1968       -2.4561      107.8358
      209.2203       36.7759      108.1310      -23.7059      107.8358       10.0081
ntw    6212.244     7257.188     2012.323  magnitude = 9762.591 m
```

By plotting the results (Fig. 3) of several runs against the error in the resulting transformation (equinoctial to cartesian, and back to equinoctial), it was hoped to determine a correlation with either the inclination or the eccentricity. Unfortunately, this proved less than decisive. The inclinations that were available from the data were not near the difficult regions of 0.0° and 180.0°, so it was not expected that this would reveal much—and it didn't! The eccentricity was clearly a factor as it got larger, but there was still a modest amount of dispersion within the data of a particular satellite. It's possible, that a correlation could be observed with additional data. However, this volume of data was not available for this study.

4.4 Covariance Propagation Results

The last test involved propagating the covariance matrices for a period of 8 days. Because the largest differences in the transformations occurred with the covariance terms, I propagated both the full covariance, as well as just the diagonal forms. To accomplish the propagation, I used the Raytheon/Geodynamics TRACE orbit determination program. This program has the "usual" force models, integrator options, as well as the ability to perform covariance analyses.

The first step was to develop a method that would suitably show the behavior of the resulting propagation. I decided to simply use the 3 position components and Root Sum Square (RSS) the results. To ensure that the error was primarily along the velocity vector direction, I transformed to the orbit plane (NTW coordinate system). This avoided any ambiguity that could exist between a Root Mean Square (RMS) and RSS value. By plotting the results, I could show simple curves representing the error growth under different conditions.



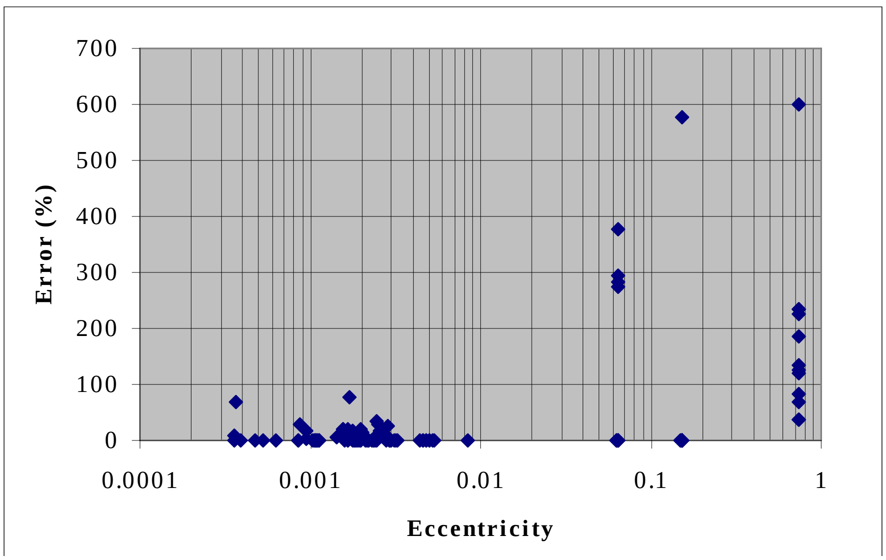
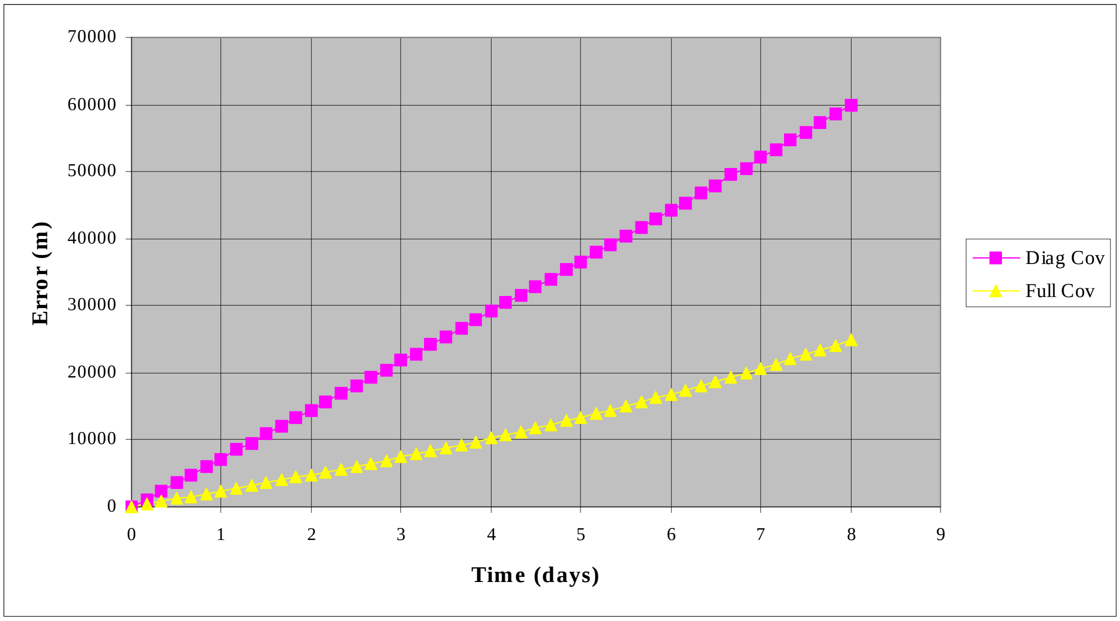


Figure 3. **Eccentricity and Inclination Variations.** This figure shows the results from several cases of satellite data. The eccentricity and inclination are used to plot the maximum error in the covariance transformation. While nothing stands out definitively, high eccentricity and inclination do produce larger errors.

For the first case, I chose the standard cartesian covariance and state. Propagating this for 8 days revealed the data shown in Fig 4.

Next, I performed a covariance propagation with the only the diagonal elements, all others being set to zero. The results are also shown in Fig 4. Notice the large difference in the errors. Many propagation accuracy studies are conducted with only the diagonal values. However, these results seem to indicate this is quite a conservative approach. Alternatively, using the full covariance matrix tended to produce much more optimistic results. It's often accepted that numerically generated covariances are overly optimistic. In fact, the covariance matrices are often multiplied by a scalar to make them appear more "realistic". The results here show that the truth is probably somewhere between both approaches, and it's very likely that the accuracy varies even with a given satellite.



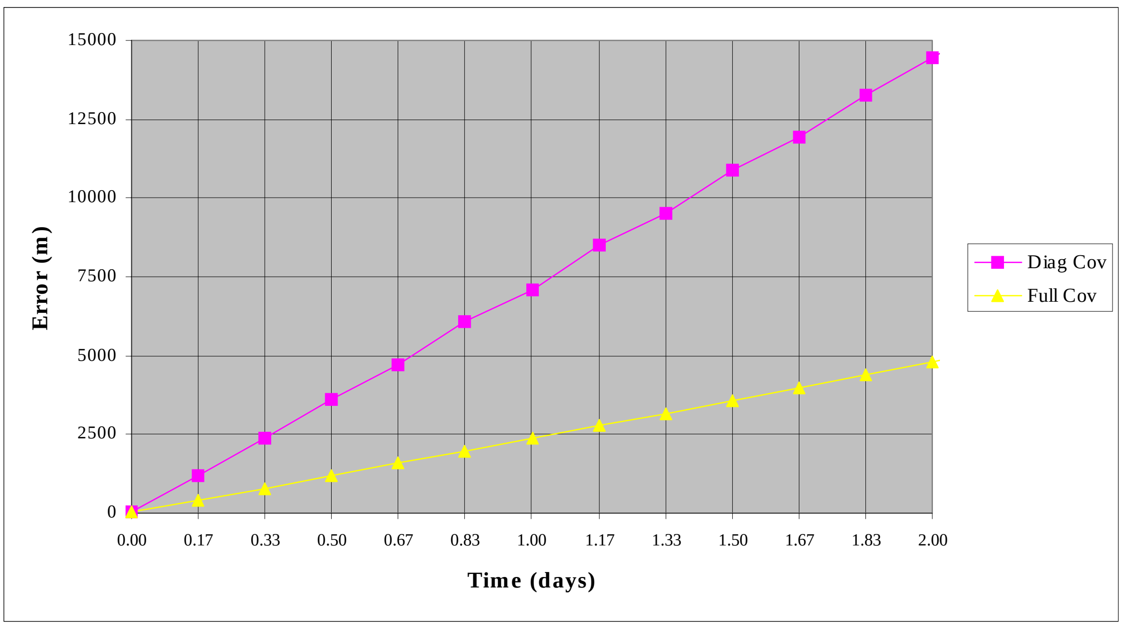


Figure 4. **Covariance Propagation.** This figure shows the propagation results including the covariance terms, and just with the diagonal terms. The bottom plot shows just the first two days, while the overall period is shown on top.

5. Conclusions

The use of covariance continues to increase in space surveillance. With accurate transformations, one can easily convert between coordinate systems and orbital state formats and choose which ever is best suited for their needs. This paper has presented an essentially mechanical approach for performing covariance transformations. Classical, cartesian, equinoctial, and flight (spherical) orbit state formats were considered, and a variety of inertial and rotating coordinate systems were presented. Sample data was given to allow reconstruction of the results. Accuracies were calculated for a variety of conditions to ensure that the transformations were correct. While test cases were presented to give an indication of the relative performance, overall trends did not readily present themselves in the data. Specifically, higher eccentricity values tended to produce less reliable results, but often, the results were quite good. Data on differing inclination values was not sufficiently distributed to be able to determine if any trends were evident. Propagation results were shown for a few conditions of the covariance matrix—complete, and just the diagonal elements. Finally, full partial derivative equations are presented in the appendix to let the reader enjoy the lengthy calculations!

6. Future Work

As often happens when writing a paper, the best ideas come a few days before the deadline for the paper. This presents the possibility to lay out future work. The present paper lends itself nicely to follow-on work, summarized below.

1. Complete the analysis of the direct transformation of equinoctial to cartesian covariances.
2. Investigate how a Kalman filter propagates the covariance data.
3. Obtain additional data for a wider variety of satellite orbits to better establish the envelope of applicability for the transformations.
4. Investigate how the covariance propagates in mean elements, vs the cartesian space results shown here. Proof of the superior behavior of the mean element characteristics would be a useful result.
5. Further investigate different satellite orbits to determine if there is a single cause to the varied performance noted in the transformations.

7. Acknowledgments

I am grateful to Jon Boers of Navy Space Command for providing some historical data to use in the preparation of this paper. The forward looking design of the Navy operational architecture made it quite easy to perform some simple studies. Paul Cefola also provided many insightful comments to improve the paper. Finally, I am grateful to Nathan Palmer who did many of the original transformations.

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Appendix

1. Partial Derivative Equations

The literature contains many references to parts of these transformations, but a complete listing is non-existant. I've combined several resources, in addition to performing numerous derivations from scratch. The sources include the Goddard Trajectory Determination Program (GTDS) math guide (Long et al, 1989), the Trajectory Analysis and Orbit Determination Program documentation, (TRACE, 1977), the Space Command TP-008 (NORAD, 1982), and other works. There are several temporary variables that will simplify the following partial derivatives.

$$\begin{aligned} d &= \vec{r} \cdot \vec{v} \\ \vec{h} &= \vec{r} \times \vec{v} \\ \vec{n} &= \hat{K} \times \vec{h} \\ \vec{A} &= \hat{h} \times \vec{r} \end{aligned}$$

Cartesian to Classical:

These partial derivatives used Eq Eq. (13) and took partials with respect to the state (position and velocity vectors). The results were as follows (See also TRACE A4-A5). For Cartesian to Classical:

$$\begin{bmatrix} & a & e & i & \Omega \\ \frac{\partial}{\partial r_I} & \frac{2 r_I a^2}{r^3} & c_1\left[r_I v^4 - d v_I v^2 - \frac{\mu r_I v^2}{r} + \frac{2\mu d v_I}{r} - \frac{2\mu d^2 r_I}{r^3}\right] & c_2\left(v_J - h_K\left(\frac{-h_J v_K + h_K v_J}{h^2}\right)\right) & c_3\left(\frac{h_J^2 v_K}{(h_I^2 + h_J^2)^{1.5}} - \frac{v_K}{\sqrt{h_I^2 + h_J^2}}\right) \\ \frac{\partial}{\partial r_J} & \frac{2 r_J a^2}{r^3} & c_1\left[r_J v^4 - d v_J v^2 - \frac{\mu r_J v^2}{r} + \frac{2\mu d v_J}{r} - \frac{2\mu d^2 r_J}{r^3}\right] & c_2\left(-v_I + h_K\left(\frac{-h_K v_I - h_I v_K}{h^2}\right)\right) & c_3\left(\frac{h_I h_J v_K}{(h_I^2 + h_J^2)^3}\right) \\ \frac{\partial}{\partial r_K} & \frac{2 r_K a^2}{r^3} & c_1\left[r_K v^4 - d v_K v^2 - \frac{\mu r_K v^2}{r} + \frac{2\mu d v_K}{r} - \frac{2\mu d^2 r_K}{r^3}\right] & c_2\left(-h_K\left(\frac{-h_I v_J + h_J v_I}{h^2}\right)\right) & c_3\left(\frac{h_I^2 v_I + h_I h_J v_J}{(h_I^2 + h_J^2)^3}\right) \\ \frac{\partial}{\partial v_I} & \frac{2 v_I a^2}{r^3} & c_1\left[2 v_I r^2 v^4 - 2\mu r v_I - v_I d^2 - r_I d v^2 + \frac{2\mu r_I d}{r}\right] & c_2\left(r_J - h_K\left(\frac{-h_J r_K + h_K r_J}{h^2}\right)\right) & c_3\left(\frac{h_J^2 r_K}{(h_I^2 + h_J^2)^3 - \frac{r_K}{\sqrt{h_I^2 + h_J^2}}}\right) \\ \frac{\partial}{\partial v_J} & \frac{2 v_J a^2}{r^3} & c_1\left[2 v_J r^2 v^4 - 2\mu r v_J - v_J d^2 - r_J d v^2 + \frac{2\mu r_J d}{r}\right] & c_2\left(-r_I + h_K\left(\frac{-h_K r_I - h_I r_K}{h^2}\right)\right) & c_3\left(\frac{h_I h_J r_K}{(h_I^2 + h_J^2)^3}\right) \\ \frac{\partial}{\partial v_K} & \frac{2 v_K a^2}{r^3} & c_1\left[2 v_K r^2 v^4 - 2\mu r v_K - v_K d^2 - r_K d v^2 + \frac{2\mu r_K d}{r}\right] & c_2\left(-h_K\left(\frac{-h_I r_J + h_J r_I}{h^2}\right)\right) & c_3\left(\frac{h_I^2 r_I + h_I h_J r_J}{(h_I^2 + h_J^2)^3}\right) \end{bmatrix}$$

$$c_1 = \frac{1}{e\mu^2} \qquad c_2 = -\frac{1}{\sqrt{h^2 - h_K^2}} \qquad c_3 = \frac{1}{\sqrt{\frac{h_I^2}{h_I^2 + h_J^2}}}$$

The remaining terms are included in the following equation.

$$
\left[
\begin{array}{ll}
 & \omega \\
\frac{\partial}{\partial r_I} & c_6\left(c_4\left[c_7\left(\frac{r_I r_K}{r^3}-\frac{v_I v_K}{\mu}\right)-2v_K h_J e_K-h_I h_J\left(\frac{v^2}{\mu}+\frac{r_I^2}{r^3}-\frac{1}{r}-\frac{v_I^2}{\mu}\right)-h_I e_I v_J-h_J h_K\left(\frac{r_I r_J}{r^3}-\frac{v_I v_J}{\mu}\right)\right.\right. \\
 & \left.\left.-h_J e_J v_J+h_K e_J v_K\right]-c_5\left[h h_I\left(\frac{r_I r_J}{r^3}-\frac{v_I v_J}{\mu}\right)+\frac{h_I e_J}{h}(v_J h_K-v_K h_J)-h h_J\left(\frac{v^2}{\mu}+\frac{r_I^2}{r^3}-\frac{1}{r}-\frac{v_I^2}{\mu}\right)+h e_I v_K-\frac{h e_I v_K}{h}(v_J h_K-v_K h_J)\right]\right) \\
\frac{\partial}{\partial r_J} & c_6\left(c_4\left[c_7\left(\frac{r_J r_K}{r^3}-\frac{v_J v_K}{\mu}\right)+2v_K h_I e_K-h_J h_K\left(\frac{v^2}{\mu}+\frac{r_J^2}{r^3}-\frac{1}{r}-\frac{v_J^2}{\mu}\right)+h_I e_I v_I-h_I h_K\left(\frac{r_I r_J}{r^3}-\frac{v_I v_J}{\mu}\right)+\right.\right. \\
 & \left.\left.h_J e_J v_I-h_K e_I v_K\right]-c_5\left[-h h_J\left(\frac{r_I r_J}{r^3}-\frac{v_I v_J}{\mu}\right)+\frac{h_I e_J}{h}(v_K h_I-v_I h_K)+h h_I\left(\frac{v^2}{\mu}+\frac{r_J^2}{r^3}-\frac{1}{r}-\frac{v_J^2}{\mu}\right)+h e_J v_K-\frac{h e_J v_K}{h}(v_J h_K-v_K h_J)\right]\right) \\
\frac{\partial}{\partial r_K} & c_6\left(c_4\left[c_7\left(\frac{v^2}{\mu}+\frac{r_K^2}{r^3}-\frac{1}{r}-\frac{v_K^2}{\mu}\right)-2v_J h_I e_K+2v_I h_J e_K-h_I h_K\left(\frac{r_I r_K}{r^3}-\frac{v_I v_K}{\mu}\right)+h_K e_I v_J-h_J h_K\left(\frac{r_J r_K}{r^3}-\frac{v_J v_K}{\mu}\right)\right.\right. \\
 & \left.\left.-h_K e_J v_I\right]-c_5\left[h h_I\left(\frac{r_J r_K}{r^3}-\frac{v_J v_K}{\mu}\right)-h e_J v_J+\frac{h_I e_J}{h}(v_I h_J-v_J h_I)-h h_J\left(\frac{r_I r_K}{r^3}-\frac{v_I v_K}{\mu}\right)+h e_I v_I-\frac{h e_I}{h}(v_I h_J-v_J h_I)\right]\right) \\
\frac{\partial}{\partial v_I} & c_6\left(c_4\left[c_7\left(\frac{2v_I r_K}{\mu}-\frac{r_I v_K}{\mu}\right)+2r_K h_J e_K-h_I h_K\left(\frac{r_I v_I}{\mu}-\frac{d}{\mu}\right)+h_I e_I r_J-h_J h_K\left(\frac{2v_I r_J}{\mu}-\frac{r_I v_J}{\mu}\right)\right.\right. \\
 & \left.\left.h_J e_J r_I-h_K e_J r_K\right]-c_5\left[h h_I\left(\frac{2v_I r_J}{\mu}-\frac{r_I v_J}{\mu}\right)+\frac{h_I e_J}{h}(r_K h_J-r_J h_K)-h h_J\left(\frac{r_I r_K}{\mu}-\frac{d}{\mu}\right)-h e_I r_K-\frac{h_J e_I}{h}(r_K h_J-r_J h_K)\right]\right) \\
\frac{\partial}{\partial v_J} & c_6\left(c_4\left[c_7\left(\frac{2v_J r_K}{\mu}-\frac{r_J v_K}{\mu}\right)-2r_K h_I e_K+-h_J h_K\left(\frac{r_J v_J}{\mu}-\frac{d}{\mu}\right)-h_I e_I r_I-h_I h_K\left(\frac{2r_I v_J}{\mu}-\frac{r_J v_I}{\mu}\right)\right.\right. \\
 & \left.\left.-h_J e_J r_I+h_K e_J r_K\right]-c_5\left[-h h_J\left(\frac{2v_J r_I}{\mu}-\frac{r_J v_I}{\mu}\right)+\frac{h_I e_J}{h}(r_I h_K-r_K h_I)-h h_I\left(\frac{r_J v_J}{\mu}-\frac{d}{\mu}\right)-h e_J r_K-\frac{h_J e_I}{h}(r_I h_K-r_K h_I)\right]\right) \\
\frac{\partial}{\partial v_K} & c_6\left(c_4\left[c_7\left(\frac{r_K v_K}{\mu}-\frac{d}{\mu}\right)+2r_J h_I e_K-2r_I h_J e_K-h_I h_K\left(\frac{2v_K r_I}{\mu}-\frac{r_K v_I}{\mu}\right)-h_K e_I r_J-h_J h_K\left(\frac{2r_J v_K}{\mu}-\frac{r_K v_J}{\mu}\right)\right.\right. \\
 & \left.\left.h_K e_J r_I\right]-c_5\left[h h_I\left(\frac{2r_J v_K}{\mu}-\frac{r_K v_J}{\mu}\right)+h e_J r_J+\frac{h_I e_J}{h}(r_J h_I-r_I h_J)-h h_J\left(\frac{2v_K r_I}{\mu}-\frac{r_K v_I}{\mu}\right)+h e_I r_I-\frac{h_J e_I}{h}(r_J h_I-r_I h_J)\right]\right)
\end{array}
\right]
$$

$$
c_4 = h(h_I e_J - h_J e_I) \qquad c_5 = h_I^2 e_K + h_J^2 e_K - h_I h_K e_I - h_J h_K e_J \qquad c_6 = \frac{1}{c_4^2 + c_5^2} \qquad c_7 = h_I^2 + h_J^2
$$

$$
\begin{bmatrix}
 & M \\
\frac{\partial}{\partial r_I} & c_9\left[c_8\left(hv_I + \frac{d}{h}(v_J h_K - v_K h_J)\right) - dh\left(2r_I v^2 - \frac{\mu r_I}{r} - 2dv_I\right)\right] \\
\frac{\partial}{\partial r_J} & c_9\left[c_8\left(hv_J + \frac{d}{h}(v_K h_I - v_I h_K)\right) - dh\left(2r_J v^2 - \frac{\mu r_J}{r} - 2dv_J\right)\right] \\
\frac{\partial}{\partial r_K} & c_9\left[c_8\left(hv_K + \frac{d}{h}(v_I h_J - v_J h_I)\right) - dh\left(2r_K v^2 - \frac{\mu r_K}{r} - 2dv_K\right)\right] \\
\frac{\partial}{\partial v_I} & c_9\left[c_8\left(hr_I + \frac{d}{h}(r_K h_J - r_J h_K)\right) - dh(2v_I r^2 - 2dr_I)\right] \\
\frac{\partial}{\partial v_J} & c_9\left[c_8\left(hr_J + \frac{d}{h}(r_I h_K - r_K h_I)\right) - dh(2v_J r^2 - 2dr_J)\right] \\
\frac{\partial}{\partial v_K} & c_9\left[c_8\left(hr_K + \frac{d}{h}(r_J h_I - r_I h_J)\right) - dh(2v_K r^2 - 2dr_K)\right]
\end{bmatrix}
$$

$$
c_8 = r^2 v^2 - \mu r - d^2 \qquad c_9 = \frac{1}{c_8^2 + d^2 h^2}
$$

For Classical to Cartesian:

The reverse process is also somewhat tedious due to the lengthy derivatives created by the transformation from IJK to PQW (See also Long, 1989, 3-58).

$$
\begin{bmatrix}
 & r_I \\
\frac{\partial}{\partial a} & c_1\left(\cos(\nu)\sin(\Omega)\sin(\omega) - \cos(\nu)\sin(\Omega)\sin(\omega)\cos(i) - \sin(\nu)\cos(\Omega)\sin(\omega) - \sin(\nu)\sin(\Omega)\cos(\omega)\cos(i)\right) \\
\frac{\partial}{\partial e} & c_2\left(\cos(\nu)\sin(\Omega)\sin(\omega) - \cos(\nu)\sin(\Omega)\sin(\omega)\cos(i) - \sin(\nu)\cos(\Omega)\sin(\omega) - \sin(\nu)\sin(\Omega)\cos(\omega)\cos(i)\right) \\
\frac{\partial}{\partial i} & ac_1\left(\sin(\Omega)\sin(i)\left(\cos(\nu)\sin(\omega) + \sin(\nu)\cos(\omega)\right)\right) \\
\frac{\partial}{\partial \Omega} & ac_1\left(-\cos(\nu)\sin(\Omega)\cos(\omega) - \cos(\nu)\cos(\Omega)\sin(\omega)\cos(i) + \sin(\nu)\sin(\Omega)\sin(\omega) - \sin(\nu)\cos(\Omega)\cos(\omega)\cos(i)\right) \\
\frac{\partial}{\partial \omega} & ac_1\left(-\cos(\nu)\cos(\Omega)\sin(\omega) - \cos(\nu)\sin(\Omega)\cos(\omega)\cos(i) - \sin(\nu)\cos(\Omega)\cos(\omega) + \sin(\nu)\sin(\Omega)\sin(\omega)\cos(i)\right) \\
\frac{\partial}{\partial M} & \\
\frac{\partial}{\partial \nu} & c_3\left(-\sin(\nu)\left(\cos(\Omega)\cos(\omega) - \sin(\Omega)\sin(\omega)\cos(i)\right) + (e + \cos(\nu))\left(-\cos(\Omega)\sin(\omega) - \sin(\Omega)\cos(\omega)\cos(i)\right)\right)
\end{bmatrix}
$$

$$
c_1 = \frac{1 - e^2}{1 + e\cos(\nu)} \qquad c_2 = -\frac{2ae + a\cos(\nu) + ae^2\cos(\nu)}{(1 + e\cos(\nu))^2} \qquad c_3 = \frac{c_1}{1 + e\cos(\nu)}
$$

$$\begin{bmatrix} & r_J \\ \frac{\partial}{\partial a} & c_1\Big(\cos(\nu)\sin(\Omega)\cos(\omega) + \cos(\nu)\cos(\Omega)\sin(\omega)\cos(i) - \sin(\nu)\sin(\Omega)\sin(\omega) + \sin(\nu)\cos(\Omega)\cos(\omega)\cos(i)\Big) \\ \frac{\partial}{\partial e} & c_2\Big(\cos(\nu)\sin(\Omega)\cos(\omega) - \cos(\nu)\cos(\Omega)\sin(\omega)\cos(i) - \sin(\nu)\sin(\Omega)\sin(\omega) + \sin(\nu)\cos(\Omega)\cos(\omega)\cos(i)\Big) \\ \frac{\partial}{\partial i} & ac_1\Big(\cos(\Omega)\sin(i)\Big)\Big(\cos(\nu)\sin(\omega) + \sin(\nu)\cos(\omega)\Big) \\ \frac{\partial}{\partial \Omega} & ac_1\Big(\cos(\nu)\cos(\Omega)\cos(\omega) - \cos(\nu)\sin(\Omega)\sin(\omega)\cos(i) - \sin(\nu)\cos(\Omega)\sin(\omega) - \sin(\nu)\sin(\Omega)\cos(\omega)\cos(i)\Big) \\ \frac{\partial}{\partial \omega} & ac_1\Big(-\cos(\nu)\sin(\Omega)\sin(\omega) + \cos(\nu)\cos(\Omega)\cos(\omega)\cos(i) - \sin(\nu)\sin(\Omega)\cos(\omega) - \sin(\nu)\cos(\Omega)\sin(\omega)\cos(i)\Big) \\ \frac{\partial}{\partial M} & \\ \frac{\partial}{\partial \nu} & c_3\Big(-\sin(\nu)(\sin(\Omega)\cos(\omega) + \cos(\Omega)\sin(\omega)\cos(i)) + (e + \cos(\nu))\Big(-\sin(\Omega)\sin(\omega) + \cos(\Omega)\cos(\omega)\cos(i)\Big)\Big) \end{bmatrix}$$

$$c_1 = \frac{1-e^2}{1+e\cos(\nu)} \qquad c_2 = -\frac{2ae + a\cos(\nu) + ae^2\cos(\nu)}{(1+e\cos(\nu))^2} \qquad c_3 = \frac{c_1}{1+e\cos(\nu)} \qquad c_4 = \sqrt{\frac{\mu}{a(1-e^2)}}$$

$$\begin{bmatrix} & r_K \\ \frac{\partial}{\partial a} & c_1\Big(\cos(\nu)\sin(\omega)\sin(i) + \sin(\nu)\cos(\omega)\sin(i)\Big) \\ \frac{\partial}{\partial e} & c_2\Big(\cos(\nu)\sin(\omega)\sin(i) + \sin(\nu)\cos(\omega)\sin(i)\Big) \\ \frac{\partial}{\partial i} & ac_1\cos(i)\Big(\cos(\nu)\sin(\omega) + \sin(\nu)\cos(\omega)\Big) \\ \frac{\partial}{\partial \Omega} & 0 \\ \frac{\partial}{\partial \omega} & ac_1\sin(i)\Big(\cos(\nu)\sin(\omega) - \sin(\nu)\cos(\omega)\Big) \\ \frac{\partial}{\partial M} & \\ \frac{\partial}{\partial \nu} & ac_3\Big(-\sin(\nu)\sin(\omega)\sin(i) + (e\cos(\nu))\Big(\cos(\omega)\sin(i)\Big)\Big) \end{bmatrix}$$

$$c_1 = \frac{1-e^2}{1+e\cos(\nu)} \qquad c_2 = -\frac{2ae + a\cos(\nu) + ae^2\cos(\nu)}{(1+e\cos(\nu))^2} \qquad c_3 = \frac{c_1}{1+e\cos(\nu)}$$

$$
\begin{bmatrix}
 & v_I \\
\frac{\partial}{\partial a} & \frac{1}{2a}c_3\left[\mathrm{SIN}(\nu)\left(\mathrm{COS}(\Omega)\,\mathrm{COS}(\omega) - \mathrm{SIN}(\Omega)\,\mathrm{SIN}(\omega)\,\mathrm{COS}(i)\right) - (e + \mathrm{COS}(\nu))\left(-\mathrm{COS}(\Omega)\,\mathrm{SIN}(\omega) - \mathrm{SIN}(\Omega)\,\mathrm{COS}((\omega)\,\mathrm{COS}(i))\right)\right] \\
\frac{\partial}{\partial e} & \frac{1}{(1-e^2)}c_3\left[-e\,\mathrm{SIN}(\nu)\left(\mathrm{COS}(\Omega)\,\mathrm{COS}(\omega) - \mathrm{SIN}(\Omega)\,\mathrm{SIN}(\omega)\,\mathrm{COS}(i)\right) + (1 + e\,\mathrm{COS}(\nu))\left(-\mathrm{COS}(\Omega)\,\mathrm{SIN}(\omega) - \mathrm{SIN}(\Omega)\,\mathrm{COS}((\omega)\,\mathrm{COS}(i))\right)\right] \\
\frac{\partial}{\partial i} & c_3\left[-\mathrm{SIN}(\nu)\,\mathrm{SIN}(\Omega)\,\mathrm{SIN}(\omega)\,\mathrm{SIN}(i) + (e + \mathrm{COS}(\nu))\left(\mathrm{SIN}(\Omega)\,\mathrm{COS}(\omega)\,\mathrm{COS}(i)\right)\right] \\
\frac{\partial}{\partial \Omega} & c_3\left[\mathrm{SIN}(\nu)\,\mathrm{SIN}(\Omega)\,\mathrm{COS}(\omega) + \mathrm{SIN}(\nu)\,\mathrm{COS}(\Omega)\,\mathrm{SIN}(\omega)\,\mathrm{COS}(i) + (e + \mathrm{COS}(\nu))\left(\mathrm{SIN}(\Omega)\,\mathrm{SIN}(\omega) - \mathrm{COS}(\Omega)\,\mathrm{COS}(\omega)\,\mathrm{COS}(i)\right)\right] \\
\frac{\partial}{\partial \omega} & c_3\left[\mathrm{SIN}(\nu)\,\mathrm{COS}(\Omega)\,\mathrm{SIN}(\omega) + \mathrm{SIN}(\nu)\,\mathrm{SIN}(\Omega)\,\mathrm{COS}(\omega)\,\mathrm{COS}(i) + (e + \mathrm{COS}(\nu))\left(-\mathrm{COS}(\Omega)\,\mathrm{COS}(\omega) + \mathrm{SIN}(\Omega)\,\mathrm{SIN}(\omega)\,\mathrm{COS}(i)\right)\right] \\
\frac{\partial}{\partial M} & \\
\frac{\partial}{\partial \nu} & c_3\left[-\mathrm{COS}(\nu)\left(\mathrm{COS}(\Omega)\,\mathrm{COS}(\omega) - \mathrm{SIN}(\Omega)\,\mathrm{SIN}(\omega)\,\mathrm{COS}(i)\right) - \mathrm{SIN}(\nu)\left(-\mathrm{COS}(\Omega)\,\mathrm{SIN}(\omega) - \mathrm{SIN}(\Omega)\,\mathrm{SIN}(\omega)\,\mathrm{COS}(i)\right)\right]
\end{bmatrix}
$$

$$
c_3 = \sqrt{\frac{\mu}{a(1-e^2)}}
$$

$$
\begin{bmatrix}
 & v_J \\
\frac{\partial}{\partial a} & \frac{1}{2a}c_3\left[\mathrm{SIN}(\nu)\left(\mathrm{SIN}(\Omega)\,\mathrm{COS}(\omega) + \mathrm{COS}(\Omega)\,\mathrm{SIN}(\omega)\,\mathrm{COS}(i)\right) - (e + \mathrm{COS}(\nu))\left(-\mathrm{SIN}(\Omega)\,\mathrm{SIN}(\omega) + \mathrm{COS}(\Omega)\,\mathrm{COS}((\omega)\,\mathrm{COS}(i))\right)\right] \\
\frac{\partial}{\partial e} & \frac{1}{(1-e^2)}c_3\left[-e\,\mathrm{SIN}(\nu)\left(\mathrm{SIN}(\Omega)\,\mathrm{COS}(\omega) + \mathrm{COS}(\Omega)\,\mathrm{SIN}(\omega)\,\mathrm{COS}(i)\right) + (1 + e\,\mathrm{COS}(\nu))\left(-\mathrm{SIN}(\Omega)\,\mathrm{SIN}(\omega) + \mathrm{COS}(\Omega)\,\mathrm{COS}((\omega)\,\mathrm{COS}(i))\right)\right] \\
\frac{\partial}{\partial i} & c_3\left[\mathrm{SIN}(\nu)\,\mathrm{COS}(\Omega)\,\mathrm{SIN}(\omega)\,\mathrm{SIN}(i) - (e + \mathrm{COS}(\nu))\left(\mathrm{COS}(\Omega)\,\mathrm{COS}(\omega)\,\mathrm{SIN}(i)\right)\right] \\
\frac{\partial}{\partial \Omega} & c_3\left[-\mathrm{SIN}(\nu)\,\mathrm{COS}(\Omega)\,\mathrm{COS}(\omega) + \mathrm{SIN}(\nu)\,\mathrm{SIN}(\Omega)\,\mathrm{SIN}(\omega)\,\mathrm{COS}(i) + (e + \mathrm{COS}(\nu))\left(-\mathrm{COS}(\Omega)\,\mathrm{SIN}(\omega) - \mathrm{SIN}(\Omega)\,\mathrm{COS}(\omega)\,\mathrm{COS}(i)\right)\right] \\
\frac{\partial}{\partial \omega} & c_3\left[\mathrm{SIN}(\nu)\,\mathrm{SIN}(\Omega)\,\mathrm{SIN}(\omega) - \mathrm{SIN}(\nu)\,\mathrm{COS}(\Omega)\,\mathrm{COS}(\omega)\,\mathrm{COS}(i) + (e + \mathrm{COS}(\nu))\left(-\mathrm{SIN}(\Omega)\,\mathrm{COS}(\omega) - \mathrm{COS}(\Omega)\,\mathrm{SIN}(\omega)\,\mathrm{COS}(i)\right)\right] \\
\frac{\partial}{\partial M} & \\
\frac{\partial}{\partial \nu} & c_3\left[-\mathrm{COS}(\nu)\left(\mathrm{SIN}(\Omega)\,\mathrm{COS}(\omega) + \mathrm{COS}(\Omega)\,\mathrm{SIN}(\omega)\,\mathrm{COS}(i)\right) - \mathrm{SIN}(\nu)(-\mathrm{SIN}(\Omega)\,\mathrm{SIN}(\omega) + \mathrm{COS}(\Omega)\,\mathrm{COS}(\omega)\,\mathrm{COS}(i))\right]
\end{bmatrix}
$$

$$
c_3 = \sqrt{\frac{\mu}{a(1-e^2)}}
$$

$$
\begin{bmatrix}
 & v_K \\
\frac{\partial}{\partial a} & \frac{1}{2a} c_3\left(\mathrm{SIN}(\nu)\,\mathrm{SIN}(\omega)\,\mathrm{SIN}(i) - (e + \mathrm{COS}(\nu))\left(\mathrm{COS}(\omega)\,\mathrm{SIN}(i)\right)\right) \\
\frac{\partial}{\partial e} & \frac{1}{(1-e^2)} c_3\left(-e\,\mathrm{SIN}(\nu)\,\mathrm{SIN}(\omega)\,\mathrm{SIN}(i) + (1 + e\,\mathrm{COS}(\nu))\left(\mathrm{COS}(\omega)\,\mathrm{SIN}(i)\right)\right) \\
\frac{\partial}{\partial i} & c_3\left(-\mathrm{SIN}(\nu)\,\mathrm{SIN}(\omega)\,\mathrm{COS}(i) + (e + \mathrm{COS}(\nu))\left(\mathrm{COS}(\omega)\,\mathrm{COS}(i)\right)\right) \\
\frac{\partial}{\partial \Omega} & 0 \\
\frac{\partial}{\partial \omega} & c_3\left(-\mathrm{SIN}(\nu)\,\mathrm{COS}(\omega)\,\mathrm{SIN}(i) - (e + \mathrm{COS}(\nu))\left(\mathrm{SIN}(\omega)\,\mathrm{SIN}(i)\right)\right) \\
\frac{\partial}{\partial M} & \\
\frac{\partial}{\partial \nu} & c_3\left(-\mathrm{COS}(\nu)\,\mathrm{SIN}(\omega)\,\mathrm{SIN}(i) - \mathrm{SIN}(\nu)\,\mathrm{COS}(\omega)\,\mathrm{SIN}(i)\right)
\end{bmatrix}
$$

$$
c_3 = \sqrt{\frac{\mu}{a(1-e^2)}}
$$

For Classical to Equinoctial:

For this conversion I show two additional rows for the anomaly options, true, mean, or time from perigee. See also TRACE A6-A8.

$$
\begin{bmatrix}
 & a_f & a_g & L(M) & L(\nu) & L(\Delta t) & n & \chi & \psi \\
\frac{\partial}{\partial a} & 0 & 0 & 0 & 0 & -\frac{3}{2}\frac{M}{a} & -\frac{3}{2a}\sqrt{\frac{\mu}{a^3}} & 0 & 0 \\
\frac{\partial}{\partial e} & \mathrm{COS}(\omega + \Omega) & \mathrm{SIN}(\omega + \Omega) & 0 & c_1 & 0 & 0 & 0 & 0 \\
\frac{\partial}{\partial i} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\mathrm{SIN}(\Omega)}{1 + \mathrm{COS}(i)} & \frac{\mathrm{COS}(\Omega)}{1 + \mathrm{COS}(i)} \\
\frac{\partial}{\partial \Omega} & -e\,\mathrm{SIN}(\omega + \Omega) & e\,\mathrm{COS}(\omega + \Omega) & 1 & 1 & 1 & 0 & \mathrm{TAN}\left(\frac{i}{2}\right)\mathrm{COS}(\Omega) & -\mathrm{TAN}\left(\frac{i}{2}\right)\mathrm{SIN}(\Omega) \\
\frac{\partial}{\partial \omega} & -e\,\mathrm{SIN}(\omega + \Omega) & e\,\mathrm{COS}(\omega + \Omega) & 1 & 1 & 1 & 0 & 0 & 0 \\
\frac{\partial}{\partial M} & 0 & 0 & 1 & & & 0 & 0 & 0 \\
\frac{\partial}{\partial \nu} & 0 & 0 & & c_2 & & 0 & 0 & 0 \\
\frac{\partial}{\partial \Delta t} & 0 & 0 & & & -n & 0 & 0 & 0
\end{bmatrix}
$$

$$
c_1 = \left(\frac{(e\,\mathrm{COS}(E) - 1)\,\mathrm{TAN}(\nu)}{1 + \left(\frac{1-e}{1+e}\right)\mathrm{TAN}\left(\frac{\nu}{2}\right)}\right)\left(\frac{1}{\sqrt{1-e^2}} + \frac{\sqrt{1-e}}{(1+e)\sqrt{1+e}}\right) - \mathrm{SIN}(E) \qquad c_2 = (1 - e\,\mathrm{COS}(E))\frac{\sqrt{\frac{1-e}{1+e}}}{\mathrm{COS}\left(\frac{\nu}{2}\right) + \left(\frac{1-e}{1+e}\right)\mathrm{SIN}\left(\frac{\nu}{2}\right)}
$$

In the partials for inclination, I've used the cosine formula, rather than the half-angle version because at small inclinations, the squared cosine term would have a larger effect on the partial derivative term. Note that an alternate form for the right ascension of the ascending node is sometimes used.

$$\frac{\partial \chi}{\partial \Omega} = \frac{\mathrm{SIN}(i)\,\mathrm{COS}(\Omega)}{1 + \mathrm{COS}(i)} \qquad \frac{\partial \psi}{\partial \Omega} = -\frac{\mathrm{SIN}(i)\,\mathrm{COS}(\Omega)}{1 + \mathrm{COS}(i)}$$

For Equinoctial to Classical:

The reverse process finds the partials for the equinoctial to classical transformation. Three columns are included for the different anomaly types.

$$\begin{bmatrix}
 & a & e & i & \Omega & \omega & M \\
\frac{\partial}{\partial a_f} & & \frac{a_f}{\sqrt{a_f^2 + a_g^2}} & 0 & 0 & \frac{-a_g}{a_f^2 + a_g^2} & \frac{a_g}{a_f^2 + a_g^2} \\
\frac{\partial}{\partial a_g} & 0 & \frac{a_g}{\sqrt{a_f^2 + a_g^2}} & 0 & 0 & \frac{a_f}{a_f^2 + a_g^2} & \frac{-a_f}{a_f^2 + a_g^2} \\
\frac{\partial}{\partial L} & 0 & 0 & 0 & 0 & 0 & 1 \\
\frac{\partial}{\partial n} & -\frac{2}{3n}\left(\frac{\mu}{n^2}\right)^{1/3} & 0 & 0 & 0 & 0 & 0 \\
\frac{\partial}{\partial \chi} & 0 & 0 & \frac{2\chi}{(1 + \chi^2 + \psi^2)\sqrt{\chi^2 + \psi^2}} & \frac{\psi}{\chi^2 + \psi^2} & \frac{-\psi}{\chi^2 + \psi^2} & 0 \\
\frac{\partial}{\partial \psi} & 0 & 0 & \frac{2\psi}{(1 + \chi^2 + \psi^2)\sqrt{\chi^2 + \psi^2}} & \frac{-\chi}{\chi^2 + \psi^2} & \frac{\chi}{\chi^2 + \psi^2} & 0
\end{bmatrix}$$

$$\begin{bmatrix}
 & \nu & \Delta t \\
\frac{\partial}{\partial a_f} & c_1\left[\frac{a_f c_2}{\sqrt{1 - e^2}} + \frac{a_f\sqrt{1 + e}\,c_2}{(1 - e^2)^{1.5}} + c_3\left(\frac{a_f \mathrm{SIN}(E)}{e} + \frac{a_g}{e^2}\right)\right] & \\
\frac{\partial}{\partial a_g} & c_1\left[\frac{a_g c_2}{\sqrt{1 - e^2}} + \frac{a_g\sqrt{1 + e}\,c_2}{(1 - e^2)^{1.5}} + c_3\left(\frac{a_g \mathrm{SIN}(E)}{e} + \frac{a_f}{e^2}\right)\right] & \\
\frac{\partial}{\partial L} & c_1 c_3 & \\
\frac{\partial}{\partial n} & 0 & \\
\frac{\partial}{\partial \chi} & 0 & \\
\frac{\partial}{\partial \psi} & 0 &
\end{bmatrix}$$

$$c_1 = \frac{1}{1 + \left(\frac{1 - e}{1 + e}\right)\mathrm{TAN}\left(\frac{E}{2}\right)} \qquad c_2 = \frac{\mathrm{TAN}\left(\frac{E}{2}\right)}{e} \qquad c_3 = \sqrt{\frac{1 - e}{1 + e}}\,\frac{1}{(1 - e\,\mathrm{COS}(E))\,\mathrm{COS}\left(\frac{E}{2}\right)}$$

For Cartesian to Flight:

The results were for cartesian to Flight,

$$
\begin{bmatrix}
 & \lambda & \phi_{gc} & \phi_{fpa} & \beta & r & v \\
\frac{\partial}{\partial r_I} & \frac{-r_J}{r_I^2 + r_J^2} & \frac{-r_I r_K}{r^2 \sqrt{r_I^2 + r_J^2}} & c_1\left[\frac{d(v_J h_K - v_K h_J)}{h} - h v_I\right] & & \frac{r_I}{r} & 0 \\
\frac{\partial}{\partial r_J} & \frac{r_I}{r_I^2 + r_J^2} & \frac{-r_J r_K}{r^2 \sqrt{r_I^2 + r_J^2}} & c_1\left[\frac{d(v_K h_I - v_I h_K)}{h} - h v_J\right] & & \frac{r_J}{r} & 0 \\
\frac{\partial}{\partial r_K} & 0 & \frac{\sqrt{r_I^2 + r_J^2}}{r^2} & c_1\left[\frac{d(v_I h_J - v_J h_I)}{h} - h v_K\right] & & \frac{r_K}{r} & 0 \\
\frac{\partial}{\partial v_I} & 0 & 0 & c_1\left[\frac{d(r_K h_J - r_J h_K)}{h} - h r_I\right] & & 0 & \frac{v_I}{v} \\
\frac{\partial}{\partial v_J} & 0 & 0 & c_1\left[\frac{d(r_I h_K - r_K h_I)}{h} - h r_J\right] & & 0 & \frac{v_J}{v} \\
\frac{\partial}{\partial v_K} & 0 & 0 & c_1\left[\frac{d(r_J h_I - r_I h_J)}{h} - h r_K\right] & & 0 & \frac{v_K}{v}
\end{bmatrix}
$$

$$c_1 = -\frac{1}{d^2 + h^2}$$

$$
\begin{bmatrix}
 & \beta \\
\frac{\partial}{\partial r_I} & c_2\left[a_K r(a_J + r_I g_K + r_I^2 v_J + r_J r_K v_K + r_J^2 v_J) - (r_I a_J - r_J a_I)\left(\frac{r_I a_K}{r} + r(r_I v_K - g_J)\right)\right] \\
\frac{\partial}{\partial r_J} & c_2\left[a_K r(-a_I + r_J g_K + r_J^2 v_I - r_I r_K v_K - r_J^2 v_J) - (r_I a_J - r_J a_I)\left(\frac{r_J a_K}{r} + r(r_J v_K - g_I)\right)\right] \\
\frac{\partial}{\partial r_K} & c_2\left[a_K r(-r_I g_I + r_I r_K v_J - r_J r_K v_I - r_J g_J) - (r_I a_J - r_J a_I)\left(\frac{r_K a_K}{r} + r(r_I v_I - r_J v_J)\right)\right] \\
\frac{\partial}{\partial v_I} & c_2[a_K r(-r_I^2 r_J - r_J r_K^2 - r_J^3) + (r_I a_J - r_J a_I)(r r_I r_K)] \\
\frac{\partial}{\partial v_J} & c_2[a_K r(r_I^3 + r_I r_K^2 + r_I r_J^2) + (r_I a_J - r_J a_I)(r r_J r_K)] \\
\frac{\partial}{\partial v_K} & c_2[(r_I a_J - r_J a_I)(r_I^2 + r_J^2)]
\end{bmatrix}
$$

$$c_2 = \frac{1}{a_K^2 r^2 + (r_I a_J - r_J a_I)^2}$$

For Flight to Cartesian:

For this operation, the vectors must be converted to both ECEF and ECI coordinates since latitude longitude values are inherently in the ECEF frame (See also Long et al., 1989, 3-45).

$$\begin{bmatrix} & r_I & r_J & r_K \\ \frac{\partial}{\partial\lambda} & 0 & 0 & 0 \\ \frac{\partial}{\partial\phi_{gc}} & 0 & 0 & 0 \\ \frac{\partial}{\partial\phi_{fpa}} & \cos(\delta)\cos(\alpha) & \cos(\delta)\sin(\alpha) & \sin(\delta) \\ \frac{\partial}{\partial\beta} & 0 & 0 & 0 \\ \frac{\partial}{\partial r} & -r\cos(\delta)\sin(\alpha) & r\cos(\delta)\cos(\alpha) & 0 \\ \frac{\partial}{\partial v} & -r\sin(\delta)\cos(\alpha) & -r\sin(\delta)\sin(\alpha) & r\cos(\delta) \end{bmatrix}$$

$$\begin{bmatrix} & v_I \\ \frac{\partial}{\partial\lambda} & v\left[\cos(\alpha)\left(\sin(\beta)\cos(\phi_{fpa})\sin(\delta)\right) - \cos(\beta)\cos(\phi_{fpa})\sin(\alpha)\right] \\ \frac{\partial}{\partial\phi_{gc}} & v\left[\cos(\alpha)\left(\cos(\beta)\sin(\phi_{fpa})\sin(\delta) + \cos(\phi_{fpa})\cos(\delta)\right) + \sin(\beta)\sin(\phi_{fpa})\sin(\alpha)\right] \\ \frac{\partial}{\partial\phi_{fpa}} & 0 \\ \frac{\partial}{\partial\beta} & \cos(\alpha)\left(-\cos(\beta)\cos(\phi_{fpa})\sin(\delta) + \sin(\phi_{fpa})\cos(\delta)\right) - \sin(\beta)\cos(\phi_{fpa})\sin(\alpha) \\ \frac{\partial}{\partial r} & v\left[-\sin(\alpha)\left(-\cos(\beta)\cos(\phi_{fpa})\sin(\delta) + \sin(\phi_{fpa})\cos(\delta)\right) - \sin(\beta)\cos(\phi_{fpa})\cos(\alpha)\right] \\ \frac{\partial}{\partial v} & v\left[\cos(\alpha)\left(-\cos(\beta)\cos(\phi_{fpa})\cos(\delta)\right) - \sin(\beta)\sin(\alpha)\right] \end{bmatrix}$$

$$\begin{bmatrix} & v_J \\ \frac{\partial}{\partial\lambda} & v\left[\sin(\alpha)\left(\sin(\beta)\cos(\phi_{fpa})\sin(\delta)\right) + \cos(\beta)\cos(\phi_{fpa})\cos(\alpha)\right] \\ \frac{\partial}{\partial\phi_{gc}} & v\left[\sin(\alpha)\left(\cos(\beta)\sin(\phi_{fpa})\sin(\delta) + \cos(\phi_{fpa})\cos(\delta)\right) - \sin(\beta)\sin(\phi_{fpa})\cos(\alpha)\right] \\ \frac{\partial}{\partial\phi_{fpa}} & 0 \\ \frac{\partial}{\partial\beta} & \sin(\alpha)\left(-\cos(\beta)\cos(\phi_{fpa})\sin(\delta) + \sin(\phi_{fpa})\cos(\delta)\right) + \sin(\beta)\cos(\phi_{fpa})\cos(\alpha) \\ \frac{\partial}{\partial r} & v\left[\cos(\alpha)\left(-\cos(\beta)\cos(\phi_{fpa})\sin(\delta) + \sin(\phi_{fpa})\cos(\delta)\right) - \sin(\beta)\cos(\phi_{fpa})\sin(\alpha)\right] \\ \frac{\partial}{\partial v} & v\left[\sin(\alpha)\left(-\cos(\beta)\cos(\phi_{fpa})\cos(\delta)\right) - \sin(\beta)\sin(\alpha)\right] \end{bmatrix}$$

$$
\begin{bmatrix}
 & v_K \\
\frac{\partial}{\partial \lambda} & -v\,\mathrm{SIN}(\beta)\,\mathrm{COS}(\phi_{fpa})\,\mathrm{COS}(\delta) \\
\frac{\partial}{\partial \phi_{gc}} & v\left(-\mathrm{COS}(\beta)\,\mathrm{COS}(\phi_{fpa})\,\mathrm{COS}(\delta) + \mathrm{COS}(\phi_{fpa})\,\mathrm{SIN}(\delta)\right) \\
\frac{\partial}{\partial \phi_{fpa}} & 0 \\
\frac{\partial}{\partial \beta} & \mathrm{COS}(\beta)\,\mathrm{COS}(\phi_{fpa})\,\mathrm{COS}(\delta) + \mathrm{SIN}(\phi_{fpa})\,\mathrm{SIN}(\delta) \\
\frac{\partial}{\partial r} & 0 \\
\frac{\partial}{\partial v} & v\left(-\mathrm{COS}(\beta)\,\mathrm{COS}(\phi_{fpa})\,\mathrm{SIN}(\delta) + \mathrm{SIN}(\phi_{fpa})\,\mathrm{COS}(\delta)\right)
\end{bmatrix}
$$

For Cartesian to Spherical:

When performing this conversion, note the similarities to the flight conversions shown earlier. Note, however, that due to the ECI and ECEF distinctions, the flight path angle and azimuth partials are different (See also Long et al, 1989, 3-41).

$$
\begin{bmatrix}
 & \alpha & \delta & \phi_{fpa} & \beta & r & v \\
\frac{\partial}{\partial r_I} & \frac{-r_J}{r_I^2 + r_J^2} & \frac{-r_I r_K}{r^2\sqrt{r_I^2 + r_J^2}} & c_1\left[\frac{r_I \dot{r}}{r} - v_I\right] & \frac{v_J\left(r v_K - r_K \dot{r}\right) - \left(\frac{r_I v_J - r_J v_I}{r}\right)\left(r_I v_K - r_K v_J + \frac{r_I r_K \dot{r}}{r}\right)}{\left(v^2 - \dot{r}^2\right)(r_I^2 + r_J^2)} & \frac{r_I}{r} & 0 \\
\frac{\partial}{\partial r_J} & \frac{r_I}{r_I^2 + r_J^2} & \frac{-r_J r_K}{r^2\sqrt{r_I^2 + r_J^2}} & c_1\left[\frac{r_J \dot{r}}{r} - v_J\right] & \frac{-v_I\left(r v_K - r_K \dot{r}\right) + \left(\frac{r_I v_J - r_J v_I}{r}\right)\left(r_J v_K - r_K v_J + \frac{r_J r_K \dot{r}}{r}\right)}{\left(v^2 - \dot{r}^2\right)(r_I^2 + r_J^2)} & \frac{r_J}{r} & 0 \\
\frac{\partial}{\partial r_K} & 0 & \frac{\sqrt{r_I^2 + r_J^2}}{r^2} & c_1\left[\frac{r_K \dot{r}}{r} - v_K\right] & \frac{\dot{r}\,(r_I v_J - r_J v_I)}{r^2\left(v^2 - \dot{r}^2\right)} & \frac{r_K}{r} & 0 \\
\frac{\partial}{\partial v_I} & 0 & 0 & c_1\left[\frac{v_I r \dot{r}}{r} - r_I\right] & \frac{-(r_J v_K - r_K v_J)}{r\left(v^2 - \dot{r}^2\right)} & 0 & \frac{v_I}{v} \\
\frac{\partial}{\partial v_J} & 0 & 0 & c_1\left[\frac{v_J r \dot{r}}{r} - r_J\right] & \frac{(r_I v_K - r_K v_I)}{r\left(v^2 - \dot{r}^2\right)} & 0 & \frac{v_J}{v} \\
\frac{\partial}{\partial v_K} & 0 & 0 & c_1\left[\frac{v_K r \dot{r}}{r} - r_K\right] & -\frac{r}{\dot{r}}\left(\frac{\dot{r}\,(r_I v_J - r_J v_I)}{r^2\left(v^2 - \dot{r}^2\right)}\right) & 0 & \frac{v_K}{v}
\end{bmatrix}
$$

$$
\dot{r} = \frac{\vec{r} \cdot \vec{v}}{r} \qquad c_1 = \frac{1}{r\sqrt{v^2 - \dot{r}^2}}
$$