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INCLUDING THE CLOSED-FORM J_2^2 EFFECT IN DSST

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A second-order closed-form semi-analytical solution of the main problem of the artificial satellite theory (J_2 contribution) consistent with the Draper Semi-analytic Satellite Theory (DSST) is presented in Delaunay variables. The short-period terms are removed using an extension of the Lie-Deprit method. The averaged equations of motion are given explicitly and transformed into the non-singular equinoctial elements. Finally, the second-order terms in the equations of motion are included in the C/C++ version of the DSST orbit propagator.

INTRODUCTION

The Draper Semi-analytic Satellite Theory (DSST) orbit propagator can be found in two forms, as an option within the Massachusetts Institute of Technology version of the Goddard Trajectory Determination System (GTDS) computer program,[1,2] and as the DSST Standalone orbit propagator package.[3–7]

The original implementations of the DSST, both in GTDS and in the Standalone versions, were done in Fortran 77 (F77). Between 2012 and 2015, the DSST was re-implemented in Java and included in the Orekit flight dynamics library.[5,6] During the same time frame, the University of La Rioja provided web access to the F77 DSST Standalone via a friendly and intuitive interface.[8] In 2016, Setty[7] extended the F77-DSST-Standalone force models, state-transition matrix, and dynamic-parameter partial derivatives to provide the F77-DSST Standalone with orbit determination capability. More recently, the migration of the F77-DSST Standalone code to C/C++ has been done at the University of La Rioja.[9,10]

The theory underlying DSST makes use of non-canonical elements and is based on the generalized method of averaging.[11–13] The Lagrange equations were used for handling the conservative terms, 50×50 gravity fields (M-daily, tesseral resonance and approximate solution of the second-order effect in J_2, J_2^2, based in eccentricity expansion only valid for circular orbits), lunar-solar point masses, and the solid Earth tides, whereas the Gauss equations for the atmospheric drag and the solar radiation pressure (SRP).

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In particular, the J_2^2 effect was formulated by Zeis[14,15] in DSST. This author developed an approximate second-order solution as a power series of the eccentricity in J_2 for mean-element and short-periodic motions. This analytical model was implemented in a computer algebra system, Macsyma. Recently, Folcik and Cefola[16] ported this code to Maxima, the open-source descendant of Macsyma, to obtain the integrant of the mean element rate expressions. Finally, the averaged process was carried out using Gauss-Kronrod numerical quadrature. These authors also performed a detailed numerical comparison between Zeis's and their numerical model and, advanced the necessity of developing an analytical closed-form model so as to improve the computational speed.

To overcome this drawback, we propose an alternative method to build a fully second-order closed-form analytical expressions for the J_2^2 contribution based on the Lie-transform method[17–21] and canonical variables consistent with the DSST orbit propagator. The main difference between the Lie transform method and the generalized method of averaging is regarded more as a matter of the algorithm than the underlying idea. In fact, both methods are connected through an integration constant.[22,23] It is worth noting that the Lie transform method allows us to obtain the transformations from osculating-to-mean and mean-to-osculating elements simultaneously.

The main characteristic observed in the application of the generalized method of averaging in the zonal problem in DSST can be found in the calculation of the zonal harmonic short-periodic generator in DSST[24] to which must be added an integration constant to guarantee that the short-period terms do not contain any long-period terms. The equivalent approaches is followed by Kozai[25] using Von Zeipel method.[26]

In this work, we develop a second-order semi-analytical theory for the main problem to remove the short-period terms from the equations of motion using Hamiltonian formalism, an extension of the Lie-Deprit method,[27] which is implemented in MathATESAT,[28] and Delaunay variables. The resultant theory is equivalent to the elimination of mean anomaly in Reference.[25] Then, the equations of motion are expressed in equinoctial elements. Finally, the second-order equations are included in DSST C.

ON THE MAIN PROBLEM AND ITS NORMALIZATION

Although extensive investigations dealing with the J_2 problem, or classically called the main problem of artificial satellite theory, have been done since Brouwer's work.[29] In this section, we present a semi-analytical theory equivalent to the first canonical transformation described in Reference[25] using an extension of the Lie-Deprit method.

In Delaunay variables (ℓ, g, h, L, G, H),[30,31] the Hamiltonian of the J_2 problem is given by

$$\mathcal{H} = \mathcal{H}_0 + \varepsilon \mathcal{H}_1, \tag{1}$$

where

$$\mathcal{H}_0 = -\frac{\mu^2}{2L^2}, \tag{2}$$

$$\mathcal{H}_1 = \frac{\mu \alpha^2}{2r^3}(3s^2 \sin^2(f+g) - 1). \tag{3}$$

μ and α are the gravitational constant and the equatorial radius of the Earth, respectively, r denotes the radial distance, s represents the sine of the inclination, f is the true anomaly and $\varepsilon = J_2$. It is worth noting that Delaunay variables are related to the classical orbital elements $(a, e, i, \omega, \Omega, M)$ by

$$
\begin{aligned}
l &= M, & L &= \sqrt{\mu a}, \\
g &= \omega, & G &= \sqrt{\mu a(1-e^2)}, \\
h &= \Omega, & H &= \sqrt{\mu a(1-e^2)}\cos i.
\end{aligned}
\tag{4}
$$

To avoid any ambiguity in the notation between Delaunay variable (ℓ, g, h, L, G, H) and the equinoctial elements (a, h, k, p, q, λ), we use Ω instead h to refer to the argument of the node in Delaunay variables.

Then, we summarize the main results of the normalization of the Hamiltonian (1) using the Extended Lie-Deprit method.[27] In this case, the Lie operator is defined from the zero order Hamiltonian $\mathcal{H}_0$ as $\mathcal{L}_{\mathcal{H}_0} = n\partial/\partial\ell$, where $n = \mu^2/L^3$ is the mean motion, whereas the solution of the homological equation $\mathcal{L}_{\mathcal{H}_0}(\mathcal{W}_n) + \mathcal{K}_n = \widetilde{\mathcal{H}}_{0\,n}$ is obtained as

$$
\mathcal{K}_n = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{H}_n \mathrm{d}\ell, \tag{5}
$$

$$
\mathcal{W}_n = \int (\widetilde{\mathcal{H}}_{0\,n} - \mathcal{K}_n)\mathrm{d}\ell + C_n, \tag{6}
$$

where C_n is an arbitrary integration function which can depend on $(\_, g, h, L, G, H)$. In this case, and in accordance with Kozai's work, C_n is chosen to remove the long-period terms contained in the generating function as

$$
C_n = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{W}_n \mathrm{d}\ell. \tag{7}
$$

After elimination the short-period terms, the second-order transformed Hamiltonian is

$$
\mathcal{K} = \mathcal{K}_0 + \mathcal{K}_1 + \frac{\varepsilon^2}{2}\mathcal{K}_2 + \mathcal{O}(\varepsilon^3), \tag{8}
$$

where

$$
\begin{aligned}
\mathcal{K}_0 &= -\frac{\mu^2}{2L^2}, \\
\mathcal{K}_i &= \eta\frac{\mu^2}{L^2}\left(\frac{\alpha^2\mu^2}{G^4}\right)^i \sum_{0\le 2k\le i} P_{i,2k}(s^2,\eta)\cos 2kg \quad \text{with } 1 \le i \le 2.
\end{aligned}
\tag{9}
$$

The coefficients $P_{i,2k}$

$$
\begin{aligned}
P_{1,0} &= \frac{3}{4}s^2 - \frac{1}{2}, \\
P_{2,0} &= -\left(\frac{15}{64}s^4 + \frac{3}{8}s^2 - \frac{3}{8}\right)\eta^2 - \left(\frac{27}{16}s^4 - \frac{9}{4}s^2 + \frac{3}{4}\right)\eta - \frac{105}{64}s^4 + \frac{15}{4}s^2 - \frac{15}{8}, \\
P_{2,2} &= \left(\frac{45}{32}s^4 - \frac{21}{16}s^2\right)(\eta^2 - 1) + \frac{3}{8}s^2\left(5s^2 - 4\right)\frac{(\eta-1)(2\eta+1)}{\eta+1},
\end{aligned}
$$

are polynomials in s and $\eta = \sqrt{1-e^2} = G/L$. On the other hand, the corresponding generating function is given to order 1 by

$$\begin{aligned} \mathcal{W}_1 &= \frac{\alpha^2\mu^2}{8\eta^3 L^3}\Big[2\left(3s^2-2\right)\phi + 2e\left(3s^2-2\right)\sin f - 3s^2\sin(2f+2g) \\ &\qquad -3es^2\sin(f+2g) - es^2\sin(3f+2g) - \frac{(\eta-1)(2\eta+1)s^2}{\eta+1}\sin 2g\Big], \end{aligned} \tag{10}$$

where $\phi = f - \ell$ is the equation of center. The expression of $\mathcal{W}_2$ is given in the Appendix A. The mean-to-osculating transformation is not provided here due to a large number of terms. The arbitrary integration functions are

$$C_1 = -\frac{\alpha^2\mu^2(\eta-1)(2\eta+1)s^2}{8\eta^3(\eta+1)L^3}\sin 2g, \tag{11}$$

$$C_2 = \frac{\alpha^4\mu^4}{256\eta^7(\eta+1)L^7}\left[2s^2\left(C_2^1 + C_2^2 s^2\right)\sin 2g - 3C_2^3 s^4 \sin 4g\right], \tag{12}$$

where the coefficients C_2^i are polynomials in η and are given by

$$\begin{aligned} C_2^1 &= 456\eta^4 + 510\eta^3 + 338\eta^2 - 382\eta - 538, \\ C_2^2 &= -540\eta^4 - 591\eta^3 - 397\eta^2 + 407\eta + 641, \\ C_2^3 &= 3(\eta-1)^2(3\eta+7). \end{aligned}$$

As usual, primes in the transformed variables have been dropped.

Finally, the partial derivatives of the Hamiltonian (8) with respect to the new variables provide the equations of motion as

$$\begin{aligned} \frac{\mathrm{d}\ell}{\mathrm{d}t} &= \frac{\partial\mathcal{K}}{\partial L} = \frac{\mu^2}{L^3} + \varepsilon\Delta_1^\ell + \frac{\varepsilon^2}{2}\Delta_2^\ell, & \frac{\mathrm{d}L}{\mathrm{d}t} &= -\frac{\partial\mathcal{K}}{\partial\ell} = 0, \\ \frac{\mathrm{d}g}{\mathrm{d}t} &= \frac{\partial\mathcal{K}}{\partial G} = \varepsilon\Delta_1^g + \frac{\varepsilon^2}{2}\Delta_2^g, & \frac{\mathrm{d}G}{\mathrm{d}t} &= -\frac{\partial\mathcal{K}}{\partial g} = \frac{\varepsilon^2}{2}\Delta_2^G, \\ \frac{\mathrm{d}\Omega}{\mathrm{d}t} &= \frac{\partial\mathcal{K}}{\partial H} = \varepsilon\Delta_1^\Omega + \frac{\varepsilon^2}{2}\Delta_2^\Omega, & \frac{\mathrm{d}H}{\mathrm{d}t} &= -\frac{\partial\mathcal{K}}{\partial h} = 0. \end{aligned} \tag{13}$$

The first-order coefficients $\Delta_1^{l,g,\Omega}$ are expressed by

$$\begin{aligned} \Delta_1^l &= -\frac{3}{4}\frac{\mu^2}{L^3}\left(\frac{\alpha^2\mu^2}{G^4}\right)(3s^2-2)\eta, \\ \Delta_1^g &= -\frac{3}{4}\frac{\mu^2}{L^3}\left(\frac{\alpha^2\mu^2}{G^4}\right)(5s^2-4), \\ \Delta_1^\Omega &= -\frac{3}{4}\frac{\mu^2}{L^3}\left(\frac{\alpha^2\mu^2}{G^4}\right)c. \end{aligned} \tag{14}$$

where c represents the cosine of the inclination.

The second-order coefficients $\Delta_2^{l,g,\Omega,G}$ are

$$\begin{aligned}
\Delta_2^{\ell} &= \frac{3}{64}\frac{\mu^2}{L^3}\left(\frac{\alpha^2\mu^2}{G^4}\right)^2 \eta\left[\bar{P}_{2,0}^{\ell} - \frac{2}{(\eta+1)^2}\bar{P}_{2,2}^{\ell}\cos 2g\right], \\
\Delta_2^{g} &= \frac{3}{64}\frac{\mu^2}{L^3}\left(\frac{\alpha^2\mu^2}{G^4}\right)^2 \left[\bar{P}_{2,0}^{g} - \frac{2}{(\eta+1)^2}\bar{P}_{2,2}^{g}\cos 2g\right], \\
\Delta_2^{\Omega} &= \frac{3}{16}\frac{\mu^2}{L^3}\left(\frac{\alpha^2\mu^2}{G^4}\right)^2 c\left[\bar{P}_{2,0}^{\Omega} - \frac{2(\eta-1)}{\eta+1}\bar{P}_{2,2}^{\Omega}\cos 2g\right], \\
\Delta_2^{G} &= \frac{3}{16}\frac{\mu^2}{L^3}\left(\frac{\alpha^2\mu^2}{G^4}\right)^2 \frac{\eta(\eta-1)}{\eta+1}s^2\bar{P}_{2,2}^{G}\sin 2g,
\end{aligned} \tag{15}$$

where the polynomials $\bar{P}_{2,i}^{l,g,\Omega,G}$ depend on s^2, η and are given by

$$\begin{aligned}
\bar{P}_{2,0}^{\ell} &= 5\left(5s^4+8s^2-8\right)\eta^2 + 16\left(2-3s^2\right)^2\eta + 15\left(7s^4-16s^2+8\right), \\
\bar{P}_{2,2}^{\ell} &= 5s^2\left(15s^2-7\right)\eta^4 + 2s^2\left(155s^2-67\right)\eta^3 + 10s^2\left(17s^2-7\right)\eta^2 \\
&\quad +30s^2\left(3-7s^2\right)\eta + 15s^2\left(3-7s^2\right), \\
\bar{P}_{2,0}^{g} &= \left(45s^4+36s^2-56\right)\eta^2 + 24\left(15s^4-22s^2+8\right)\eta \\
&\quad +5\left(77s^4-172s^2+88\right), \\
\bar{P}_{2,2}^{g} &= \left(135s^4-158s^2+28\right)\eta^4 + \left(670s^4-732s^2+120\right)\eta^3 + 2\left(55s^4\right. \\
&\quad \left.-66s^2+16\right)\eta^2 - 10\left(77s^4-82s^2+12\right)\eta - 5\left(77s^4-82s^2+12\right), \\
\bar{P}_{2,0}^{\Omega} &= \left(5s^2+4\right)\eta^2 + \left(36s^2-24\right)\eta + 5\left(7s^2-8\right), \\
\bar{P}_{2,2}^{\Omega} &= \left(15s^2-7\right)\eta^2 + \left(70s^2-30\right)\eta + 5\left(7s^2-3\right), \\
\bar{P}_{2,2}^{G} &= \left(15s^2-14\right)\eta^2 + \left(70s^2-60\right)\eta + 5\left(7s^2-6\right).
\end{aligned} \tag{16}$$

MEAN EQUINOCTIAL VARIATIONAL EQUATIONS

In this section, we derive the mean equinoctial variational equations from Eqs. (13). The equinoctial elements[32] are defined in terms of the orbital elements as follows:

$$\begin{aligned}
a &= a, & p &= \tan^{\mathrm{I}}(i/2)\sin\Omega, \\
h &= e\sin(\omega+\mathrm{I}\Omega), & q &= \tan^{\mathrm{I}}(i/2)\cos\Omega, \\
k &= e\cos(\omega+\mathrm{I}\Omega), & \lambda &= M+\omega+\mathrm{I}\Omega,
\end{aligned} \tag{17}$$

where I is called the retrograde factor. I takes the value 1 for the direct equinoctial elements and -1 for the retrograde equinoctial elements.

To derive the equinoctial equation of motion, we start by differentiating Eqs. (17):

$$\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{\mathrm{d}a}{\mathrm{d}t}, \\
\frac{\mathrm{d}h}{\mathrm{d}t} &= \frac{h}{e}\frac{\mathrm{d}e}{\mathrm{d}t} + k\left(\frac{\mathrm{d}\omega}{\mathrm{d}t} + \mathrm{I}\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right), \\
\frac{\mathrm{d}k}{\mathrm{d}t} &= \frac{k}{e}\frac{\mathrm{d}e}{\mathrm{d}t} - h\left(\frac{\mathrm{d}\omega}{\mathrm{d}t} + \mathrm{I}\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right), \\
\frac{\mathrm{d}p}{\mathrm{d}t} &= \frac{1}{2}\mathrm{I}\sec^2(i/2)\tan^{\mathrm{I}-1}(i/2)\sin\Omega\frac{\mathrm{d}i}{\mathrm{d}t} + q\frac{\mathrm{d}\Omega}{\mathrm{d}t}, \\
\frac{\mathrm{d}q}{\mathrm{d}t} &= \frac{1}{2}\mathrm{I}\sec^2(i/2)\tan^{\mathrm{I}-1}(i/2)\cos\Omega\frac{\mathrm{d}i}{\mathrm{d}t} - p\frac{\mathrm{d}\Omega}{\mathrm{d}t}, \\
\frac{\mathrm{d}\lambda}{\mathrm{d}t} &= \frac{\mathrm{d}M}{\mathrm{d}t} + \frac{\mathrm{d}\omega}{\mathrm{d}t} + \mathrm{I}\frac{\mathrm{d}\Omega}{\mathrm{d}t}.
\end{aligned} \tag{18}$$

The derivatives on the right-hand sides of Eqs. (18) are obtained in a straightforward manner by differentiating Eqs. (4) as

$$\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{2L}{\mu}\frac{\mathrm{d}L}{\mathrm{d}t}, \\
\frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{1}{eL}\left((1-e^2)\frac{\mathrm{d}L}{\mathrm{d}t} - \sqrt{1-e^2}\frac{\mathrm{d}G}{\mathrm{d}t}\right), \\
\frac{\mathrm{d}i}{\mathrm{d}t} &= \frac{1}{G\sin i}\left(\cos i\frac{\mathrm{d}G}{\mathrm{d}t} - \frac{\mathrm{d}H}{\mathrm{d}t}\right), \\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{\mathrm{d}g}{\mathrm{d}t},\ \frac{\mathrm{d}M}{\mathrm{d}t} = \frac{\mathrm{d}\ell}{\mathrm{d}t}.
\end{aligned} \tag{19}$$

Then, considering mean elements in Eqs. (18), (19) and using the Hamilton's equations Eq. (13), after some algebraic manipulation, we obtain the mean equinoctial variational equations as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \sum_{k=0}^{2}\frac{\varepsilon^k}{k!}\bar{\Delta}_k^{\sigma}, \tag{20}$$

where $\mathrm{d}a/\mathrm{d}t = 0$ and σ represents the mean equinoctial elements (h, k, p, q, λ). It is worth noting that Eqs. (20) depend on Delaunay variables g, L, G and the orbital elements e, i through η, s, c.

Finally, we will obtain the expressions of Eqs (20) in equinoctial element taking into account Eqs.

(17) and the transformation from the equinoctial elements to the orbital elements given by

$$
\begin{aligned}
a &= a, \\
e &= \sqrt{h^2+k^2}, \\
i &= \pi\left(\frac{1-\mathrm{I}}{2}\right) + 2\mathrm{I}\arctan\sqrt{p^2+q^2}, \\
\sin\Omega &= \frac{p}{\sqrt{p^2+q^2}}, \\
\cos\Omega &= \frac{q}{\sqrt{p^2+q^2}}, \\
\omega &= \zeta - \mathrm{I}\Omega, \\
M &= \lambda - \zeta,
\end{aligned} \tag{21}
$$

where ζ is an auxiliary angle, which is defined by:

$$
\begin{aligned}
\sin\zeta &= \frac{h}{\sqrt{h^2+k^2}}, \\
\cos\zeta &= \frac{k}{\sqrt{h^2+k^2}}.
\end{aligned} \tag{22}
$$

The zero-order terms in equinoctial elements are $\bar{\Delta}_0^h = \bar{\Delta}_0^k = \bar{\Delta}_0^p = \bar{\Delta}_0^q = 0$ and $\bar{\Delta}_0^\lambda = n$ where $n = \sqrt{\mu/a^3}$ is the mean motion in equinoctial elements.

The first-order terms only depend on the momenta L, G and η, s, c allowing us to obtain the Δ_1^σ terms in a straightforward manner as

$$
\begin{aligned}
\Delta_1^h &= -\mathcal{C}_1 k\chi, \\
\Delta_1^k &= \mathcal{C}_1 h\chi, \\
\Delta_1^p &= -2\mathcal{C}_1 cq, \\
\Delta_1^q &= 2\mathcal{C}_1 cp, \\
\Delta_1^\lambda &= -\mathcal{C}_1\left[(3\eta+5)s^2 + 2\mathrm{I}c - 2(\eta+2)\right],
\end{aligned} \tag{23}
$$

where $\mathcal{C}_1 = 3\alpha^2 n/(4a^2\eta^4)$ and $\chi = 5s^2 + 2\mathrm{I}c - 4$. The expressions of η, c and s in equinoctial elements are

$$
\eta = \sqrt{1-h^2-k^2}, \quad c = \frac{1-p^2-q^2}{1+p^2+q^2}, \quad s = \frac{2\sqrt{p^2+q^2}}{1+p^2+q^2}.
$$

As can be seen, Eqs. (23) are valid for both direct and retrograde equinoctial elements and agree with known results.[14,33]

The expression of the second-order terms to equinoctial elements are not immediate. The second-order Hamilton's equations introduce the contribution of the argument of the perigee g through terms in $\sin 2g$ and $\cos 2g$. Using Eqs. (17), these trigonometric expressions are converted to equinoctial elements and yield

$$
\begin{aligned}
\sin 2g &= \frac{2(hq - \mathrm{I}kp)(\mathrm{I}hp + kq)}{(h^2+k^2)(p^2+q^2)}, \\
\cos 2g &= \frac{\left(h^2-k^2\right)\left(p^2-q^2\right) + 4\mathrm{I}hkpq}{(h^2+k^2)(p^2+q^2)}.
\end{aligned} \tag{24}
$$

After a laborious process of simplification, we obtained that the Δ_2^σ terms were not entirely non-singular, these depended on the factor $(p^2+q^2)^{-2I}$. Therefore, it was not possible to derive a unique set of expressions valid for both direct and retrograde orbits.

For the case of direct orbits ($I=1$), we obtain the following non-singular expressions

$$
\begin{aligned}
\Delta_2^h &= \mathcal{C}_2 \sum_{m_1=0}^{1}\sum_{m_2=0}^{3}\sum_{m_3=0}^{8}\sum_{m_4=0}^{8} \mathrm{P}_h^{(m_1,m_2,m_3,m_4)} h^{m_1} k^{m_2} p^{m_3} q^{m_4}, \\
\Delta_2^k &= \mathcal{C}_2 \sum_{m_1=0}^{3}\sum_{m_2=0}^{1}\sum_{m_3=0}^{8}\sum_{m_4=0}^{8} \mathrm{P}_k^{(m_1,m_2,m_3,m_4)} h^{m_1} k^{m_2} p^{m_3} q^{m_4}, \\
\Delta_2^p &= \mathcal{C}_2^* \sum_{m_1=0}^{2}\sum_{m_2=0}^{2}\sum_{m_3=0}^{8}\sum_{m_4=0}^{9} \mathrm{P}_p^{(m_1,m_2,m_3,m_4)} h^{m_1} k^{m_2} p^{m_3} q^{m_4}, \\
\Delta_2^q &= \mathcal{C}_2^* \sum_{m_1=0}^{2}\sum_{m_2=0}^{2}\sum_{m_3=0}^{9}\sum_{m_4=0}^{8} \mathrm{P}_q^{(m_1,m_2,m_3,m_4)} h^{m_1} k^{m_2} p^{m_3} q^{m_4}, \\
\Delta_2^\lambda &= \mathcal{C}_2 \sum_{m_1=0}^{2}\sum_{m_2=0}^{2}\sum_{m_3=0}^{8}\sum_{m_4=0}^{8} \mathrm{P}_\lambda^{(m_1,m_2,m_3,m_4)} h^{m_1} k^{m_2} p^{m_3} q^{m_4},
\end{aligned} \tag{25}
$$

where

$$
\begin{aligned}
\mathcal{C}_2 &= \frac{3\alpha^4 n}{8a^4\eta^8(\eta+1)^3\left(p^2+q^2+1\right)^4}, \\
\mathcal{C}_2^* &= \frac{3\alpha^4 n\left(p^2+q^2-1\right)}{8a^4\eta^8(\eta+1)^2\left(p^2+q^2+1\right)^3},
\end{aligned} \tag{26}
$$

and the coefficients $\mathrm{P}_\sigma^{(m_1,m_2,m_3,m_4)}$ are polynomial in η. It is worth noting that direct orbits can include the range of inclination $[0,\pi)$.

For the case of retrograde orbits ($I=-1$) and $i=\pi$ we have that $p=q=0$ and take the limits when $p\to 0$ and $q\to 0$ and we obtain

$$
\begin{aligned}
\Delta_2^h &= \mathcal{C}_2' k[q_3(\eta) + 4q_2(\eta)h^2], \\
\Delta_2^k &= \mathcal{C}_2' h[q_5(\eta) + 4q_2(\eta)k^2], \\
\Delta_2^p &= 0, \\
\Delta_2^q &= 0, \\
\Delta_2^\lambda &= -\mathcal{C}_2'[q_1(\eta)q_4(\eta) + 4q_2(\eta)k^2],
\end{aligned} \tag{27}
$$

$$\mathcal{C}_2' = \frac{3\alpha^4 n}{8a^4\eta^8(\eta+1)^2}$$

Finally, the second-order terms have been tested reproducing the solution provided by Zeis.[14] This approach has been derived from the beginning considering the first-order approximation in eccentricity of Eq. (3) which yields

$$\mathcal{H}_2 = -\frac{3\alpha^4\mu^6\left(19s^4-30s^2+12\right)}{16L^{10}} + \mathcal{O}\left(e^2\right), \tag{28}$$

Following the same process, we obtain

$$\begin{aligned}
\Delta_2^h &= \mathcal{C}_2^Z k\left(19s^2-15\right)\left(s^2+\mathrm{I}c-1\right), \\
\Delta_2^k &= -\mathcal{C}_2^Z h\left(19s^2-15\right)\left(s^2+\mathrm{I}c-1\right), \\
\Delta_2^p &= \mathcal{C}_2^Z cq\left(19s^2-15\right), \\
\Delta_2^q &= -\mathcal{C}_2^Z cp\left(19s^2-15\right), \\
\Delta_2^\lambda &= \frac{\mathcal{C}_2^Z}{2}\left[2\left(19s^2-15\right)\left(s^2+\mathrm{I}c-1\right)+5\left(19s^4-30s^2+12\right)\eta\right],
\end{aligned} \tag{29}$$

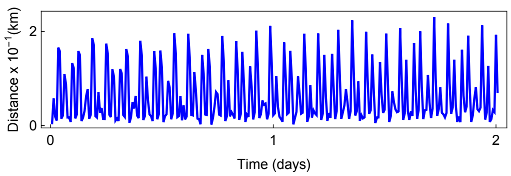
where $\mathcal{C}_2^Z = 3\alpha^4 n/(4a^4\eta)$. In this simplified case, the expressions do not include any singularity and are valid for both direct and retrograde orbits and agree with known results.[14,33]

NUMERICAL EXPERIMENTS

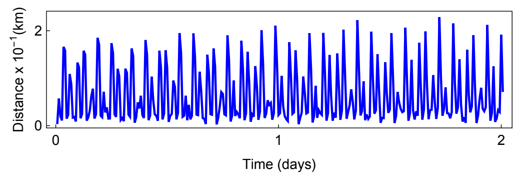
Finally, we performed a comparison between the full second-order semi-analytical theory and the approximation given by Zeis.[14] The test model considered was the non-transformed equations of motion. A highly accurate eighth order Runge-Kutta method[34] was used to integrate the transformed and non-transformed equations of motion. Two orbits with different semi-major axis and eccentricity values were propagated for a time span of two days. The distance, along-track, cross-track, and radial errors between the two semi-analytical theories and the test model were plotted. The epoch osculating elements adopted for this comparison were fixed $M = 0°$, $\omega = 0°$, $\Omega = 0°$, and $i = 30°$. In the first case, the orbit had 200 km perigee height by 210 km apogee height and, in the second, 200 km perigee height by 42000 km apogee height. The first test represents a near circular orbit while the second a highly eccentric orbit. Both of the orbital cases were considered by Folcik in 2012.

The transformed full second order and Zeis equations of motion are expressed in equinoctial elements, while the direct and inverse transformations (osculating-to-mean and mean-to-osculating) are still in Delaunay variables. The integration of both semi-analytical theories starts with transforming the initial osculating elements to the initial mean elements. Then the mean equations of motion are integrated and, finally, transformation from mean elements to osculating elements is applied for each integration step. In the case of Zeis approximation, we truncate both transformations, osculating-to-mean and mean-to-osculating, and the mean equations of motion up to first order in eccentricity. It is worth noting that no differential adjustment of constants or initial conditions was performed.

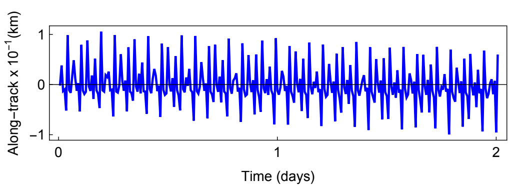
Figure 1 shows the distance, along-track, cross-track, and radial errors between the full second order and Zeis semi-analytical theories and the test model for a near-circular orbit with $e = 0.00076$. For the case of the full second-order theory, the maximum distance, along-track, cross-track, and radial errors are 0.23, 0.11, 0.19 and 0.03 km, respectively, whereas in the case of Zeis solution are 0.23, 0.11, 0.19 and 0.04 km, respectively. As can be seen, the errors are similar when the eccentricity is very small. It is worth noting that although the direct and inverse transformations have not been expressed in non-singular variables yet, they do not present any numerical problem.



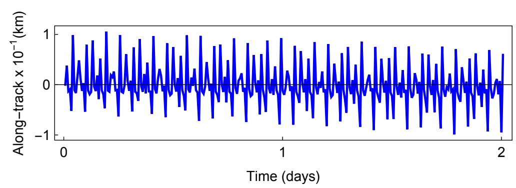
(a) Distance error (km). Full second order..



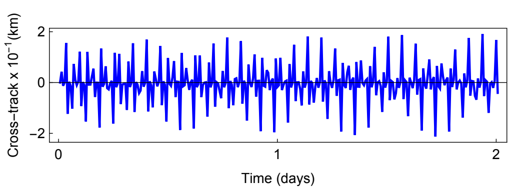
(b) Distance error (km). Zeis approximation.



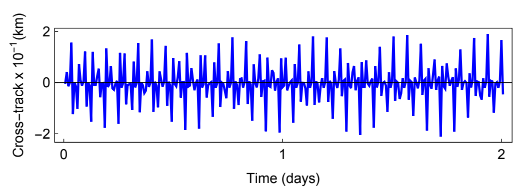
(c) Along-track error (km). Full second order..



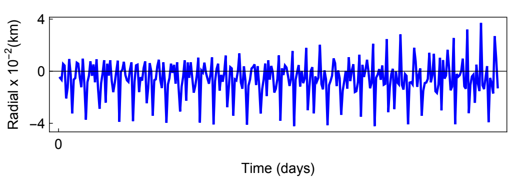
(d) Along-track error (km). Zeis approximation.



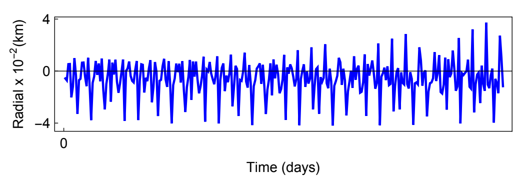
(e) Cross-track error (km). Full second order..



(f) Cross-track error (km). Zeis approximation.



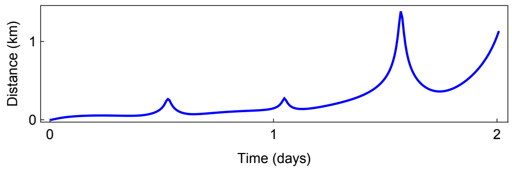
(g) Radial error (km). Full second order..



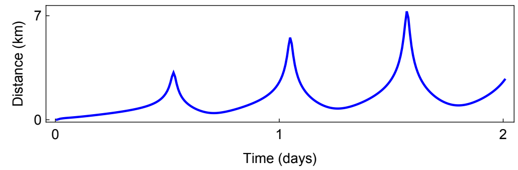
(h) Radial error (km). Zeis approximation.

Figure 1. Distance, along-track, cross-track, and radial errors between the full second-order and Zeis semi-analytical theories and the test model. $M = 0°$**,** $\omega = 0°$**,** $\Omega = 0°$**,** $i = 30°$ **whereas, the orbit has 200 km perigee height by 210 km apogee height, that is,** $a = 6583.14$ **km and** $e = 0.00076$**.**

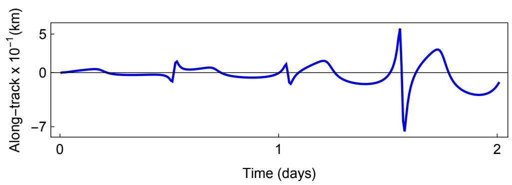
Figure 2 shows the distance, along-track, cross-track, and radial errors between the full second order and Zeis semi-analytical theories and the test model for a highly eccentric orbit with $e = 0.76$. For the case of the full second-order theory, the maximum distance, along-track, cross-track, and radial errors are 1.38, 0.57, 0.45 and 0.56 km, respectively, whereas in the case of Zeis solution are 7.29, 4.00, 6.52 and 4.01 km, respectively. The distance error in Zeis solution increases about 6 km respect to full second-order solution. We note that the proportionality between the full solution and Zeis solution results in Figure 2 is like the proportionality between the numerical quadrature and Zeis solution results given in Folcik, 2012.



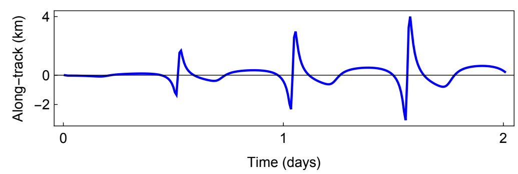
(a) Distance error (km). Full second order.



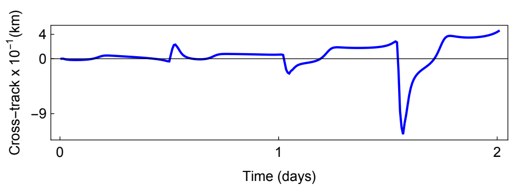
(b) Distance error (km). Zeis approximation.



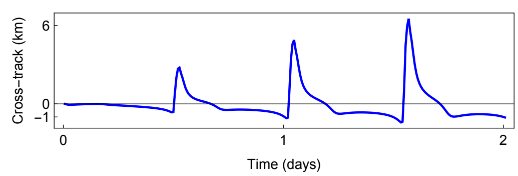
(c) Along-track error (km). Full second order..



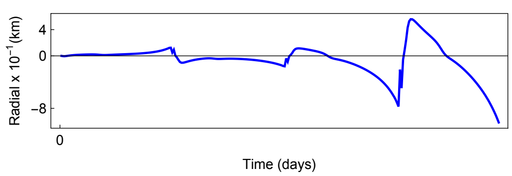
(d) Along-track error (km). Zeis approximation.



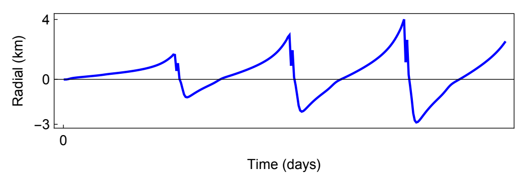
(e) Cross-track error (km). Full second order..



(f) Cross-track error (km). Zeis approximation.



(g) Radial error (km). Full second order..



(h) Radial error (km). Zeis approximation.

Figure 2. Distance, along-track, cross-track, and radial errors between the full second-order and Zeis semi-analytical theories and the test model. $M = 0°$**,** $\omega = 0°$**,** $\Omega = 0°$**,** $i = 30°$ **whereas, the orbit had 200 km perigee height by 42000 km apogee height, that is,** $a = 27478.1$ **km and** $e = 0.76$**.**

CONCLUSION AND FUTURE WORK

In this paper, a second order closed-form semi-analytical solution of the J_2 problem has been developed. An extension of the Lie-Deprit method and Delaunay variables is used to remove the short-period terms from the equations of motion using the Hamiltonian formalism. This semi-analytical theory has been implemented using MathATESAT.

The averaged equations of motion are given explicitly and transformed into the non-singular equinoctial elements. This semi-analytical theory is equivalent to others in which Lagrange Planetary equations, as well as General Averaging Method and equinoctial elements are used, and hence, consistent with the Draper Semi-analytic Satellite Theory. However, it has been proven that using Lie-Deprit method allows us to generate a semi-analytical theory easier than using the General Averaging Method.

Additionally, the second-order terms have been tested reproducing the solution provided by Zeis, validated numerically, and included in DSST C.

Finally, we are currently working on expressing the second order mean-to-osculating transformation in equinoctial elements and calculating the partial derivatives necessary for an orbit determination system.

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APPENDIX A: $\mathcal{W}_2$

The second-order generating function is

$$
\begin{aligned}
\mathcal{W}_2 \;=\; & \frac{\alpha^4\mu^4}{L^7\eta^7}\left[-\frac{3}{64}\phi\left(5\mathcal{P}_7 s^4+8\mathcal{P}_8 s^2-8\mathcal{P}_6\right)\right. \\
&+\frac{1}{128\mathcal{P}_1}s^2\left(\mathcal{P}_{34}s^2+\mathcal{P}_{33}\right)\sin 2g+\frac{3}{256\mathcal{P}_1}s^4\mathcal{P}_2^2\mathcal{P}_3\sin 4g \\
&+\frac{3}{32\mathcal{P}_1}\mathcal{P}_2\phi\left(5\mathcal{P}_{13}s^4-2\mathcal{P}_{17}s^2\right)\cos 2g \\
&+\frac{3}{128e}\mathcal{P}_2\left(\mathcal{P}_{29}s^4-8\mathcal{P}_{25}s^2+8\mathcal{P}_{19}\right)\sin f \\
&-\frac{3}{128\mathcal{P}_1}\mathcal{P}_2\left(\mathcal{P}_{26}s^4+8\mathcal{P}_{14}s^2-8\mathcal{P}_{12}\right)\sin 2f \\
&+\frac{1}{256\mathcal{P}_1 e}\mathcal{P}_2^2\left(\mathcal{P}_{28}s^4+16\mathcal{P}_{11}s^2-16\mathcal{P}_1^2\right)\sin 3f \\
&-\frac{9}{128\mathcal{P}_1}s^4\mathcal{P}_2^2\sin 4f+\frac{3}{256e}s^4\mathcal{P}_2^3\sin 5f \\
&-\frac{3}{32\mathcal{P}_1 e}\mathcal{P}_2^2\left(\mathcal{P}_{10}s^4-2\mathcal{P}_9 s^2\right)\sin(f-2g) \\
&-\frac{3}{64e}s^2\mathcal{P}_2\left(\mathcal{P}_{31}s^2+\mathcal{P}_{30}\right)\sin(f+2g) \\
&+\frac{3}{256e}s^4\mathcal{P}_2^3\sin(f-4g)+\frac{3}{256\mathcal{P}_1 e}s^4\mathcal{P}_2^2\mathcal{P}_{21}\sin(f+4g) \\
&+\frac{3}{64\mathcal{P}_1}\mathcal{P}_2^2\mathcal{S}_2\sin(2f-2g)+\frac{3}{64}s^2\left(\mathcal{P}_{23}s^2+\mathcal{P}_{24}\right)\sin(2f+2g) \\
&+\frac{15}{256\mathcal{P}_1}s^4\mathcal{P}_2\mathcal{P}_{16}\sin(2f+4g) \\
&-\frac{1}{128e}\mathcal{S}_2\mathcal{P}_2^3\sin(3f-2g)-\frac{1}{32e}s^2\mathcal{P}_2\left(\mathcal{P}_{27}s^2-2\mathcal{P}_{18}\right)\sin(3f+2g) \\
&+\frac{15}{256e}s^4\mathcal{P}_2\mathcal{P}_{15}\sin(3f+4g) \\
&-\frac{3}{128}s^2\mathcal{P}_2\left(\mathcal{P}_5 s^2-2\mathcal{P}_4\right)\sin(4f+2g)+\frac{3}{256}s^4\mathcal{P}_{20}\sin(4f+4g) \\
&-\frac{3}{128}e\mathcal{S}_2\mathcal{P}_2\sin(5f+2g)+\frac{3}{256e}s^4\mathcal{P}_{32}\sin(5f+4g) \\
&-\frac{3}{256}e^2s^4\sin(6f+4g) \\
&\left.+\frac{9}{16}e\mathcal{S}_1\cos(f+2g)+\frac{9}{16}\mathcal{S}_1\cos(2f+2g)+\frac{3}{16}e\mathcal{S}_1\cos(3f+2g)\right],
\end{aligned}
\tag{30}
$$

where $\mathcal{S}_1 = s^2\left(5s^2-4\right)\phi$, $\mathcal{S}_2 = s^2\left(3s^2-2\right)$ and $\mathcal{P}_i$ represent polynomials in η given by

$$
\begin{aligned}
\mathcal{P}_1 &= \eta+1, \\
\mathcal{P}_2 &= \eta-1, \\
\mathcal{P}_3 &= 3\eta+7, \\
\mathcal{P}_4 &= 5\eta+11, \\
\mathcal{P}_5 &= 13\eta+31, \\
\mathcal{P}_6 &= \eta^2-5, \\
\mathcal{P}_7 &= \eta^2+7, \\
\mathcal{P}_8 &= \eta^2-10, \\
\mathcal{P}_9 &= \eta(3\eta+4), \\
\mathcal{P}_{10} &= 7\eta^2+9\eta-1, \\
\mathcal{P}_{11} &= \eta^2+3\eta+2, \\
\mathcal{P}_{12} &= \eta^2+4\eta+3, \\
\mathcal{P}_{13} &= 3\eta^2+10\eta+5, \\
\mathcal{P}_{14} &= \eta^2+6\eta+6, \\
\mathcal{P}_{15} &= \eta^2+2\eta-7, \\
\mathcal{P}_{16} &= \eta^2-2\eta-7, \\
\mathcal{P}_{17} &= 7\eta^2+22\eta+11, \\
\mathcal{P}_{18} &= \eta^2+8\eta+14, \\
\mathcal{P}_{19} &= \eta^2+12\eta+15, \\
\mathcal{P}_{20} &= \eta^2-18\eta+21, \\
\mathcal{P}_{21} &= \eta^2+18\eta+21, \\
\mathcal{P}_{22} &= 2\eta^2+9\eta+23, \\
\mathcal{P}_{23} &= 2\eta^2+9\eta-23, \\
\mathcal{P}_{24} &= -4\eta^2-6\eta+26, \\
\mathcal{P}_{25} &= 5\eta^2+21\eta+30, \\
\mathcal{P}_{26} &= 5\eta^2+4\eta-31, \\
\mathcal{P}_{27} &= 3\eta^2+25\eta+43, \\
\mathcal{P}_{28} &= 25\eta^2-6\eta-59, \\
\mathcal{P}_{29} &= 49\eta^2+37\eta+112, \\
\mathcal{P}_{30} &= -20\eta^2+102\eta+126, \\
\mathcal{P}_{31} &= 26\eta^2-109\eta-145, \\
\mathcal{P}_{32} &= 3\eta^3-7\eta^2-3\eta+7, \\
\mathcal{P}_{33} &= -456\eta^4-510\eta^3-338\eta^2+382\eta+538, \\
\mathcal{P}_{34} &= 540\eta^4+591\eta^3+397\eta^2-407\eta-641.
\end{aligned}
\tag{31}
$$

(32)

REFERENCES

[1] J. O. Cappellari, A. C. Long, C. E. Velez, and A. J. Fuchs, "Goddard Trajectory Determination System (GTDS)," Tech. Rep. CSC/TR-89/6021, Goddard Space Flight Center, 1989.

[2] Z. Folcik and P. Cefola, "Very Long Arc Timing Coefficient and Solar Lunar Planetary Ephemeris Files and Application," *Proceedings 29th AAS/AIAA Space Flight Mechanics Meeting*, Ka'anapali, HI, USA, January 13-17 2019. Paper AAS 19-401.

[3] L. W. Early, "A portable orbit generator using semianalytical satellite theory," *Proceedings 1986 AIAA/AAS Astrodynamics Conference*, Williamsburg, VA, USA, American Institute of Aeronautics and Astronautics, August 1986. Paper AIAA 86-2164-CP.

[4] J. G. Neelon, Jr., P. J. Cefola, and R. J. Proulx, "Current development of the Draper Semianalytical Satellite Theory standalone orbit propagator package," *Advances in the Astronautical Sciences*, Vol. 97, 1998, pp. 2037–2052. Paper AAS 97-731.

[5] P. J. Cefola, B. Bentley, L. Maisonobe, P. Parraud, R. Di-Constanzo, and Z. Folcik, "Verification of the Orekit Java implementation of the Draper Semi-Analytical Satellite Theory," *Advances in the Astronautical Sciences*, Vol. 148, 2013, pp. 3079–3110. Paper AAS 13-398.

[6] P. J. Cefola, Z. Folcik, R. Di-Costanzo, N. Bernard, S. Setty, and J. F. San-Juan, "Revisiting the DSST standalone orbit propagator," *Advances in the Astronautical Sciences*, Vol. 152, 2014, pp. 2891–2914. Paper AAS 14-411.

[7] S. J. Setty, P. J. Cefola, O. Montenbruck, and H. Fiedler, "Application of Semi-analytical Satellite Theory orbit propagator to orbit determination for space object catalog maintenance," *Advances in Space Research*, Vol. 57, May 2016, pp. 2218–2233, 10.1016/j.asr.2016.02.028.

[8] J. F. San-Juan, M. Lara, R. López, L. M. López, B. Weeden, and P. J. Cefola, "Allocation of DSST in the new implementation of $\mathcal{A}strody^{Web}_{Tools}$," *Proceedings Advanced Maui Optical and Space Surveillance Technologies Conference, AMOS 2012*, Maui, HI, USA, Maui Economic Development Board, September 2012, pp. 346–362.

[9] J. F. San-Juan, R. López, R. Suanes, I. Pérez, S. J. Setty, and P. J. Cefola, "Migration of the DSST Standalone to C/C++," *Advances in the Astronautical Sciences*, Vol. 160, 2017, pp. 2419–2437. Paper AAS 17-369.

[10] J. F. San-Juan, R. López, R. Suanes, I. Pérez, S. J. Setty, and P. J. Cefola, "Validation of DSST C/C++ against original Fortran version: integration test," *Proceedings of the AIAA Scitech 2020 Forum*, Orlando, FO, USA, American Institute of Aeronautics and Astronautics, January 2020. Paper AIAA 20-0955, 10.2514/6.2020-0955.

[11] N. M. Krylov and N. N. Bogoliubov, *Introduction to Non-linear Mechanics*. Princeton, NJ, USA: Princeton University Press, 1943.

[12] N. N. Bogoliubov and Y. A. Mitropolsky, *Asymptotic Methods in the Theory of Non-linear Oscillations*. Translated from the second revised Russian edition. International Monographs on Advanced Mathematics and Physics, Hindustan Publishing Corp., Delhi, India; Gordon and Breach Science Publishers, New York, NY, USA, 1961.

[13] W. D. McClain, "A recursively formulated first-order semianalytic artificial satellite theory based on the generalized method of averaging. Volume 1: The generalized method of averaging applied to the artificial satellite problem," Tech. Rep. CSC/TR-77/6010, Computer Sciences Corporation, June 1977.

[14] E. G. Zeis, "A computerized algebraic utility for the construction of nonsingular satellite theories," Master's thesis, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, MA, USA, September 1978.

[15] J. Fischer, "The evolution of highly eccentric orbits," Master's thesis, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, MA, USA, June 1998.

[16] Z. Folcik and P. J. Cefola, "A general solution to the second order J2 contribution in a mean element semianalytical satellite theory," *Proceedings Advanced Maui Optical and Space Surveillance Technologies Conference, AMOS 2012*, Maui, HI, USA, Maui Economic Development Board, September 2012.

[17] G.-i. Hori, "Theory of general perturbations with unspecified canonical variables," *Publications of the Astronomical Society of Japan*, Vol. 18, No. 4, 1966, pp. 287–296.

[18] A. Deprit, "Canonical transformations depending on a small parameter," *Celestial Mechanics*, Vol. 1, March 1969, pp. 12–30, 10.1007/BF01230629.

[19] J. Henrard, "On a perturbation theory using Lie transforms," *Celestial Mechanics*, Vol. 3, March 1970, pp. 107–120, 10.1007/BF01230436.

[20] A. A. Kamel, "Perturbation method in the theory of nonlinear oscillations," *Celestial Mechanics*, Vol. 3, March 1970, pp. 90–106, 10.1007/BF01230435.

[21] G.-i. Hori, "Theory of general perturbations for non-canonical systems," *Publications of the Astronomical Society of Japan*, Vol. 23, 1971, pp. 567–587.

[22] J. A. Morrison, "Generalized method of averaging and the von Zeipel method," *Methods in Astrodynamics and Celestial Mechanics* (R. L. Duncombe and V. G. Szebehely, eds.), Vol. 17 of *Progress in Astronautics and Aeronautics*, pp. 117–138, American Institute of Aeronautics and Astronautics, January 1966, 10.2514/5.9781600864919.0117.0138.

[23] H. Shniad, "The equivalence of von Zeipel mappings and Lie transforms," *Celestial Mechanics*, Vol. 2, March 1970, pp. 114–120, 10.1007/BF01230455.

[24] M. Slutsky, "Zonal harmonic short-periodic model developed for the Precision Orbit Propagation (POP) contract," Intralab Memorandum PL-016-81-MS, Charles Stark Draper Laboratory, Cambridge, MA, USA, November 1981.

[25] Y. Kozai, "Second-order solution of artificial satellite theory without air drag," *The Astronomical Journal*, Vol. 67, September 1962, pp. 446–461, 10.1086/108753.

[26] H. v. Zeipel, "Recherches sur le mouvement des petites planètes," *Arkiv för Matematik, Astronomi och Fysik*, Vol. 11, No. 1-2, 1916.

[27] J. F. San-Juan, R. López, I. Pérez, and M. San-Martín, "A note about certain arbitrariness in the solution of the homological equation in Deprit's method," *Mathematical Problems in Engineering*, Vol. 2015, 2015, p. 10 pages. Article ID 982857, 10.1155/2015/982857.

[28] J. F. San-Juan, L. M. López, and R. López, "MathATESAT: A symbolic-numeric environment in Astrodynamics and Celestial Mechanics," *Lecture Notes in Computer Science*, Vol. 6783, No. 2, 2011, pp. 436–449, 10.1007/978-3-642-21887-3_34.

[29] D. Brouwer, "Solution of the problem of artificial satellite theory without drag," *The Astronomical Journal*, Vol. 64, November 1959, pp. 378–397, 10.1086/107958.

[30] C. Delaunay, *Théorie du Mouvement de la Lune. Tome Premier*, Vol. XXVIII of *Extrait des Mémoires de l'Académie des Sciences de l'Institut Impérial de France*. Paris: Mallet-Bachelier, 1860.

[31] D. E. Chang and J. E. Marsden, "Geometric derivation of the Delaunay variables and geometric phases," *Celestial Mechanics and Dynamical Astronomy*, Vol. 86, June 2003, pp. 185–208, 10.1023/A:1024174702036.

[32] R. Broucke and P. Cefola, "On the equinoctial orbit elements," *Celestial Mechanics*, Vol. 1, May 1972, pp. 303–310, 10.1007/BF01228432.

[33] D. A. Danielson, B. Neta, and L. W. Early, "Semianalytic Satellite Theory (SST): mathematical algorithms," Tech. Rep. NPS-MA-94-001, Naval Postgraduate School, Monterey, CA, USA, January 1994.

[34] J. R. Dormand and P. J. Prince, "Practical Runge-Kutta processes," *SIAM Journal on Scientific and Statistical Computing*, Vol. 10, No. 5, 1989, pp. 977–989, 10.1137/0910057.