

THE SQUARE-ROOT UNSCENTED KALMAN FILTER FOR STATE AND PARAMETER-ESTIMATION

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ABSTRACT

Over the last 20-30 years, the *extended Kalman filter* (EKF) has become the algorithm of choice in numerous nonlinear estimation and machine learning applications. These include estimating the state of a nonlinear dynamic system as well estimating parameters for nonlinear system identification (*e.g.*, learning the weights of a neural network). The EKF applies the standard linear Kalman filter methodology to a linearization of the true nonlinear system. This approach is sub-optimal, and can easily lead to divergence. Julier et al. [1] proposed the *unscented Kalman filter* (UKF) as a derivative-free alternative to the extended Kalman filter in the framework of state-estimation. This was extended to parameter-estimation by Wan and van der Merwe [2, 3]. The UKF consistently outperforms the EKF in terms of prediction and estimation error, at an equal computational complexity of $\mathcal{O}(L^3)^1$ for general state-space problems. When the EKF is applied to parameter-estimation, the special form of the state-space equations allows for an $\mathcal{O}(L^2)$ implementation. This paper introduces the *square-root unscented Kalman filter* (SR-UKF) which is also $\mathcal{O}(L^3)$ for general state-estimation and $\mathcal{O}(L^2)$ for parameter estimation (note the original formulation of the UKF for parameter-estimation was $\mathcal{O}(L^3)$). In addition, the square-root forms have the added benefit of numerical stability and guaranteed positive semi-definiteness of the state covariances.

1. INTRODUCTION

The EKF has been applied extensively to the field of nonlinear estimation for both *state-estimation* and *parameter-estimation*. The basic framework for the EKF (and the UKF) involves estimation of the state of a discrete-time nonlinear dynamic system,

$$\mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k, \mathbf{u}_k) + \mathbf{v}_k \quad (1)$$

$$\mathbf{y}_k = \mathbf{H}(\mathbf{x}_k) + \mathbf{n}_k, \quad (2)$$

where \mathbf{x}_k represent the unobserved state of the system, \mathbf{u}_k is a known exogenous input, and \mathbf{y}_k is the observed measurement signal. The *process* noise \mathbf{v}_k drives the dynamic system, and the *observation* noise is given by \mathbf{n}_k . The EKF involves the recursive estimation of the mean and covariance of the state under a Gaussian assumption.

In contrast, parameter-estimation, sometimes referred to as system identification, involves determining a nonlinear mapping

This work was sponsored in part by NSF under grant IRI-9712346, ECS-0083106, and DARPA under grant F33615-98-C-3516.

¹ L is the dimension of the state variable.

$\mathbf{y}_k = \mathbf{G}(\mathbf{x}_k, \mathbf{w})$, where \mathbf{x}_k is the input, \mathbf{y}_k is the output, and the nonlinear map, $\mathbf{G}(\cdot)$, is parameterized by the vector \mathbf{w} . Typically, a training set is provided with sample pairs consisting of known input and desired outputs, $\{\mathbf{x}_k, \mathbf{d}_k\}$. The error of the machine is defined as $\mathbf{e}_k = \mathbf{d}_k - \mathbf{G}(\mathbf{x}_k, \mathbf{w})$, and the goal of learning involves solving for the parameters \mathbf{w} in order to minimize the expectation of some given function of the error. While a number of optimization approaches exist (*e.g.*, gradient descent and Quasi-Newton methods), parameters can be efficiently estimated on-line by writing a new state-space representation

$$\mathbf{w}_{k+1} = \mathbf{w}_k + \mathbf{r}_k \quad (3)$$

$$\mathbf{d}_k = \mathbf{G}(\mathbf{x}_k, \mathbf{w}_k) + \mathbf{e}_k, \quad (4)$$

where the parameters \mathbf{w}_k correspond to a stationary process with identity state transition matrix, driven by process noise \mathbf{r}_k (the choice of variance determines convergence and tracking performance). The output \mathbf{d}_k corresponds to a nonlinear observation on \mathbf{w}_k . The EKF can then be applied directly as an efficient “second-order” technique for learning the parameters [4].

2. THE UNSCENTED KALMAN FILTER

The inherent flaws of the EKF are due to its linearization approach for calculating the mean and covariance of a random variable which undergoes a nonlinear transformation. As shown in shown in [1, 2, 3], the UKF addresses these flaws by utilizing a deterministic “sampling” approach to calculate mean and covariance terms. Essentially, $2L + 1$, *sigma* points (L is the state dimension), are chosen based on a square-root decomposition of the prior covariance. These sigma points are propagated through the true nonlinearity, without approximation, and then a weighted mean and covariance is taken. A simple illustration of the approach is shown in Figure 1 for a 2-dimensional system: the left plot shows the true mean and covariance propagation using Monte-Carlo sampling; the center plots show the results using a linearization approach as would be done in the EKF; the right plots show the performance of the new “sampling” approach (note only 5 sigma points are required). This approach results in approximations that are accurate to the third order (Taylor series expansion) for Gaussian inputs for all nonlinearities. For non-Gaussian inputs, approximations are accurate to at least the second-order [1]. In contrast, the linearization approach of the EKF results only in first order accuracy.

The full UKF involves the recursive application of this “sampling” approach to the state-space equations. The standard UKF implementation is given in Algorithm 2.1 for state-estimation, and uses the following variable definitions: $\{W_i\}$ is a set of scalar weights ($W_0^{(m)} = \lambda/(L + \lambda)$, $W_0^{(c)} = \lambda/(L + \lambda) + (1 - \alpha^2 + \beta)$

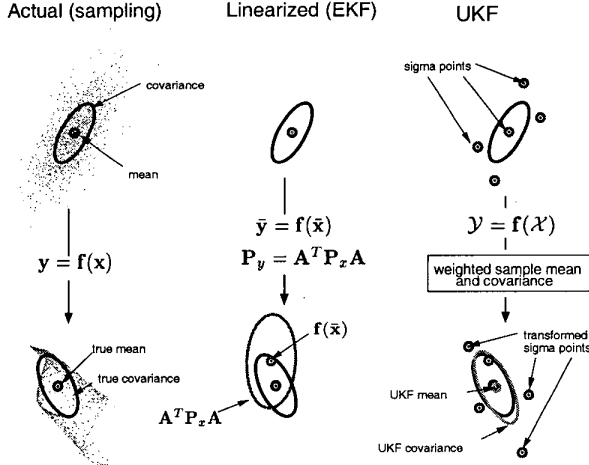


Figure 1: Example of mean and covariance propagation. a) actual, b) first-order linearization (EKF), c) new "sampling" approach (UKF).

, $W_i^{(m)} = W_i^{(c)} = 1/\{2(L + \lambda)\}$ $i = 1, \dots, 2L$. $\lambda = L(\alpha^2 - 1)$ and $\eta = \sqrt{L + \lambda}$ are scaling parameters. The constant α determines the spread of the sigma points around $\hat{\mathbf{x}}$ and is usually set to $1e - 4 \leq \alpha \leq 1$. β is used to incorporate prior knowledge of the distribution of \mathbf{x} (for Gaussian distributions, $\beta = 2$ is optimal). Also note that we define the linear algebra operation of adding a column vector to a matrix, i.e. $\mathbf{A} \pm \mathbf{u}$ as the addition of the vector to each column of the matrix. The superior performance of the UKF over the EKF has been demonstrated in a number of applications [1, 2, 3]. Furthermore, unlike the EKF, no explicit derivatives (i.e., Jacobians or Hessians) need to be calculated.

3. EFFICIENT SQUARE-ROOT IMPLEMENTATION

The most computationally expensive operation in the UKF corresponds to calculating the new set of sigma points at each time update. This requires taking a matrix square-root of the state covariance matrix², $\mathbf{P} \in \mathbb{R}^{L \times L}$, given by $\mathbf{S}\mathbf{S}^T = \mathbf{P}$. An efficient implementation using a Cholesky factorization requires in general $\mathcal{O}(L^3/6)$ computations [5]. While the square-root of \mathbf{P} is an integral part of the UKF, it is still the full covariance \mathbf{P} which is recursively updated. In the SR-UKF implementation, \mathbf{S} will be propagated directly, avoiding the need to refactorize at each time step. The algorithm will in general still be $\mathcal{O}(L^3)$, but with improved numerical properties similar to those of standard square-root Kalman filters [6]. Furthermore, for the special state-space formulation of parameter-estimation, an $\mathcal{O}(L^2)$ implementation becomes possible.

The square-root form of the UKF makes use of three powerful linear algebra techniques³, *QR decomposition*, *Cholesky factor updating* and *efficient least squares*, which we briefly review below:

- *QR decomposition*. The QR decomposition or factorization of a matrix $\mathbf{A} \in \mathbb{R}^{L \times N}$ is given by, $\mathbf{A}^T = \mathbf{Q}\mathbf{R}$, where $\mathbf{Q} \in \mathbb{R}^{N \times N}$ is orthogonal, $\mathbf{R} \in \mathbb{R}^{N \times L}$ is upper triangular and $N \geq L$. The upper triangular part of \mathbf{R} , $\bar{\mathbf{R}}$, is the transpose of the Cholesky factor of $\mathbf{P} = \mathbf{A}\mathbf{A}^T$, i.e.,

²For notational clarity, the time index k has been omitted.

³See [5] for theoretical and implementation details.

Initialize with:

$$\hat{\mathbf{x}}_0 = \mathbb{E}[\mathbf{x}_0] \quad \mathbf{P}_0 = \mathbb{E}[(\mathbf{x}_0 - \hat{\mathbf{x}}_0)(\mathbf{x}_0 - \hat{\mathbf{x}}_0)^T] \quad (5)$$

For $k \in \{1, \dots, \infty\}$,

Calculate sigma points:

$$\mathcal{X}_{k-1} = [\hat{\mathbf{x}}_{k-1} \quad \hat{\mathbf{x}}_{k-1} + \eta\sqrt{\mathbf{P}_{k-1}} \quad \hat{\mathbf{x}}_{k-1} - \eta\sqrt{\mathbf{P}_{k-1}}] \quad (6)$$

Time update:

$$\mathcal{X}_{k|k-1} = \mathbf{F}[\mathcal{X}_{k-1}, \mathbf{u}_{k-1}] \quad (7)$$

$$\hat{\mathbf{x}}_k^- = \sum_{i=0}^{2L} W_i^{(m)} \mathcal{X}_{i,k|k-1} \quad (8)$$

$$\mathbf{P}_k^- = \sum_{i=0}^{2L} W_i^{(c)} [\mathcal{X}_{i,k|k-1} - \hat{\mathbf{x}}_k^-][\mathcal{X}_{i,k|k-1} - \hat{\mathbf{x}}_k^-]^T + \mathbf{R}^v \quad (9)$$

$$\mathbf{y}_{k|k-1} = \mathbf{H}[\mathcal{X}_{k|k-1}]$$

$$\hat{\mathbf{y}}_k^- = \sum_{i=0}^{2L} W_i^{(m)} \mathbf{y}_{i,k|k-1} \quad (10)$$

Measurement update equations:

$$\mathbf{P}_{\hat{\mathbf{y}}_k \hat{\mathbf{y}}_k} = \sum_{i=0}^{2L} W_i^{(c)} [\mathbf{y}_{i,k|k-1} - \hat{\mathbf{y}}_k^-][\mathbf{y}_{i,k|k-1} - \hat{\mathbf{y}}_k^-]^T + \mathbf{R}^n$$

$$\mathbf{P}_{\mathbf{x}_k \mathbf{y}_k} = \sum_{i=0}^{2L} W_i^{(c)} [\mathcal{X}_{i,k|k-1} - \hat{\mathbf{x}}_k^-][\mathbf{y}_{i,k|k-1} - \hat{\mathbf{y}}_k^-]^T \quad (11)$$

$$\mathcal{K}_k = \mathbf{P}_{\mathbf{x}_k \mathbf{y}_k} \mathbf{P}_{\hat{\mathbf{y}}_k \hat{\mathbf{y}}_k}^{-1} \quad (12)$$

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + \mathcal{K}_k (\mathbf{y}_k - \hat{\mathbf{y}}_k^-) \quad (13)$$

$$\mathbf{P}_k = \mathbf{P}_k^- - \mathcal{K}_k \mathbf{P}_{\hat{\mathbf{y}}_k \hat{\mathbf{y}}_k} \mathcal{K}_k^T \quad (14)$$

where \mathbf{R}^v =process noise cov., \mathbf{R}^n =measurement noise cov.

Algorithm 2.1: Standard UKF algorithm.

$\bar{\mathbf{R}} = \mathbf{S}^T$, such that $\bar{\mathbf{R}}^T \bar{\mathbf{R}} = \mathbf{A}\mathbf{A}^T$. We use the shorthand notation $\text{qr}\{\cdot\}$ to denote a QR decomposition of a matrix where only $\bar{\mathbf{R}}$ is returned. The computational complexity of a QR decomposition is $\mathcal{O}(NL^2)$. Note that performing a Cholesky factorization directly on $\mathbf{P} = \mathbf{A}\mathbf{A}^T$ is $\mathcal{O}(L^3/6)$ plus $\mathcal{O}(NL^2)$ to form $\mathbf{A}\mathbf{A}^T$.

- *Cholesky factor updating*. If \mathbf{S} is the original Cholesky factor of $\mathbf{P} = \mathbf{A}\mathbf{A}^T$, then the Cholesky factor of the rank-1 update (or downdate) $\mathbf{P} \pm \sqrt{\nu}\mathbf{u}\mathbf{u}^T$ is denoted as $\mathbf{S} = \text{cholupdate}\{\mathbf{S}, \mathbf{u}, \pm\nu\}$. If \mathbf{u} is a matrix and not a vector, then the result is M consecutive updates of the Cholesky factor using the M columns of \mathbf{u} . This algorithm (available in Matlab as `cholupdate`) is only $\mathcal{O}(L^2)$ per update.
- *Efficient least squares*. The solution to the equation $(\mathbf{A}\mathbf{A}^T)\mathbf{x} = \mathbf{A}^T\mathbf{b}$ also corresponds to the solution of the overdetermined least squares problem $\mathbf{A}\mathbf{x} = \mathbf{b}$. This can be solved efficiently using a QR decomposition with pivoting (implemented in Matlab's `\'` operator).

The complete specification of the new square-root filters is given in Algorithm 3.1 for state-estimation and 3.2 for parameter-

estimation. Below we describe the key parts of the square-root algorithms, and how they contrast with the standard implementations.

Square-Root State-Estimation: As in the original UKF, the filter is initialized by calculating the matrix square-root of the state covariance once via a Cholesky factorization (Eqn. 16). However, the propagated and updated Cholesky factor is then used in subsequent iterations to directly form the sigma points. In Eqn. 20 the *time-update* of the Cholesky factor, \mathbf{S}^- , is calculated using a QR decomposition of the compound matrix containing the weighted propagated sigma points and the matrix square-root of the additive process noise covariance. The subsequent Cholesky update (or dowdate) in Eqn. 21 is necessary since the zero'th weight, $W_0^{(c)}$, may be negative. These two steps replace the *time-update* of \mathbf{P}^- in Eqn. 9, and is also $\mathcal{O}(L^3)$.

The same two-step approach is applied to the calculation of the Cholesky factor, $\mathbf{S}_{\tilde{y}}$, of the observation-error covariance in Eqns. 24 and 25. This step is $\mathcal{O}(LM^2)$, where M is the observation dimension. In contrast to the way the Kalman gain is calculated in the standard UKF (see Eqn. 12), we now use two nested inverse (or *least squares*) solutions to the following expansion of Eqn. 12, $\mathcal{K}_k(\mathbf{S}_{\tilde{y}_k} \mathbf{S}_{\tilde{y}_k}^T) = \mathbf{P}_{\mathbf{x}_k \mathbf{y}_k}$. Since $\mathbf{S}_{\tilde{y}}$ is square and triangular, efficient "back-substitutions" can be used to solve for \mathcal{K}_k directly without the need for a matrix inversion.

Finally, the posterior measurement update of the Cholesky factor of the state covariance is calculated in Eqn. 29 by applying M sequential Cholesky dowdates to \mathbf{S}_k^- . The dowdate vectors are the columns of $\mathbf{U} = \mathcal{K}_k \mathbf{S}_{\tilde{y}_k}$. This replaces the posterior update of \mathbf{P}_k in Eqn. 14, and is also $\mathcal{O}(LM^2)$.

Square-Root Parameter-Estimation: The parameter-estimation algorithm follows a similar framework as that of the state-estimation square-root UKF. However, an $\mathcal{O}(ML^2)$ algorithm, as opposed to $\mathcal{O}(L^3)$, is possible by taking advantage of the *linear* state transition function. Specifically, the time-update of the state covariance is given simply by $\mathbf{P}_{\mathbf{w}_k}^- = \mathbf{P}_{\mathbf{w}_{k-1}} + \mathbf{R}_{k-1}^r$. Now, if we apply an exponential weighting on past data⁴, the process noise covariance is given by $\mathbf{R}_k^r = (\gamma^{-1} - 1)\mathbf{P}_{\mathbf{w}_k}$, and the time update of the state covariance becomes,

$$\mathbf{P}_{\mathbf{w}_k}^- = \mathbf{P}_{\mathbf{w}_{k-1}} + (\gamma^{-1} - 1)\mathbf{P}_{\mathbf{w}_{k-1}} = \gamma^{-1}\mathbf{P}_{\mathbf{w}_{k-1}}. \quad (15)$$

This translates readily into the factored form, $\mathbf{S}_{\mathbf{w}_k}^- = \gamma^{-1/2}\mathbf{S}_{\mathbf{w}_{k-1}}$ (see Eqn. 32), and avoids the costly $\mathcal{O}(L^3)$ QR and Cholesky based updates necessary in the state-estimation filter.

4. EXPERIMENTAL RESULTS

The improvement in error performance of the UKF over that of the EKF for both state and parameter-estimation is well documented [1, 2, 3]. The focus of this section will be to simply verify the equivalent error performance of the UKF and SR-UKF, and show the reduction in computational cost achieved by the SR-UKF for parameter-estimation. Figure 2 shows the superior performance of UKF and SR-UKF compared to that of the EKF on estimating the Mackey-Glass-30 chaotic time series corrupted by additive white noise (3dB SNR). The error performance of the SR-UKF and UKF are indistinguishable and are both superior to the EKF. The computational complexity of all three filters are of the same order but

⁴This is identical to the approach used in weighted recursive least squares (W-RLS). γ is a scalar weighting factor chosen to be slightly less than 1, i.e. $\gamma = 0.9995$.

Initialize with:

$$\hat{\mathbf{x}}_0 = \mathbb{E}[\mathbf{x}_0] \quad \mathbf{S}_0 = \text{chol} \left\{ \mathbb{E}[(\mathbf{x}_0 - \hat{\mathbf{x}}_0)(\mathbf{x}_0 - \hat{\mathbf{x}}_0)^T] \right\} \quad (16)$$

For $k \in \{1, \dots, \infty\}$,

Sigma point calculation and time update:

$$\mathcal{X}_{k-1} = [\hat{\mathbf{x}}_{k-1} \quad \hat{\mathbf{x}}_{k-1} + \eta \mathbf{S}_k \quad \hat{\mathbf{x}}_{k-1} - \eta \mathbf{S}_k] \quad (17)$$

$$\mathcal{X}_{k|k-1} = \mathbf{F}[\mathcal{X}_{k-1}, \mathbf{u}_{k-1}] \quad (18)$$

$$\hat{\mathbf{x}}_k^- = \sum_{i=0}^{2L} W_i^{(m)} \mathcal{X}_{i,k|k-1} \quad (19)$$

$$\mathbf{S}_k^- = \text{qr} \left\{ \left[\sqrt{W_1^{(c)}} (\mathcal{X}_{1:2L,k|k-1} - \hat{\mathbf{x}}_k^-) \quad \sqrt{\mathbf{R}^v} \right] \right\} \quad (20)$$

$$\mathbf{S}_k^- = \text{cholupdate} \left\{ \mathbf{S}_k^-, \mathcal{X}_{0,k} - \hat{\mathbf{x}}_k^-, W_0^{(c)} \right\} \quad (21)$$

$$\mathcal{Y}_{k|k-1} = \mathbf{H}[\mathcal{X}_{k|k-1}] \quad (22)$$

$$\hat{\mathbf{y}}_k^- = \sum_{i=0}^{2L} W_i^{(m)} \mathcal{Y}_{i,k|k-1} \quad (23)$$

Measurement update equations:

$$\mathbf{S}_{\tilde{y}_k} = \text{qr} \left\{ \left[\sqrt{W_1^{(c)}} [\mathcal{Y}_{1:2L,k} - \hat{\mathbf{y}}_k] \quad \sqrt{\mathbf{R}^n} \right] \right\} \quad (24)$$

$$\mathbf{S}_{\tilde{y}_k} = \text{cholupdate} \left\{ \mathbf{S}_{\tilde{y}_k}, \mathcal{Y}_{0,k} - \hat{\mathbf{y}}_k, W_0^{(c)} \right\} \quad (25)$$

$$\mathbf{P}_{\mathbf{x}_k \mathbf{y}_k} = \sum_{i=0}^{2L} W_i^{(c)} [\mathcal{X}_{i,k|k-1} - \hat{\mathbf{x}}_k^-][\mathcal{Y}_{i,k|k-1} - \hat{\mathbf{y}}_k^-]^T \quad (26)$$

$$\mathcal{K}_k = (\mathbf{P}_{\mathbf{x}_k \mathbf{y}_k} / \mathbf{S}_{\tilde{y}_k}^T) / \mathbf{S}_{\tilde{y}_k} \quad (27)$$

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + \mathcal{K}_k (\mathbf{y}_k - \hat{\mathbf{y}}_k^-) \quad (28)$$

$$\mathbf{U} = \mathcal{K}_k \mathbf{S}_{\tilde{y}_k} \quad (28)$$

$$\mathbf{S}_k = \text{cholupdate} \left\{ \mathbf{S}_k^-, \mathbf{U}, -1 \right\} \quad (29)$$

where \mathbf{R}^v =process noise cov., \mathbf{R}^n =measurement noise cov.

Algorithm 3.1: Square-Root UKF for state-estimation.

the SR-UKF is about 20% faster than the UKF and about 10% faster than the EKF.

The next experiment shows the reduction in computational cost achieved by the square-root unscented Kalman filters and how that compares to the computational complexity of the EKF for parameter-estimation. For this experiment, we use an EKF, UKF and SR-UKF to train a 2-12-2 MLP neural network on the well known *Mackay-Robot-Arm*⁵ benchmark problem of mapping the joint angles of a robot arm to the Cartesian coordinates of the hand. The learning curves (mean square error (MSE) vs. learning epoch) of the different filters are shown in Figure 3. Figure 4 shows how the computational complexity of the different filters scale as a function of the number of parameters (weights in neural network). While the standard UKF is $\mathcal{O}(L^3)$, both the EKF and SR-UKF are $\mathcal{O}(L^2)$.

⁵<http://wol.ra.phy.cam.ac.uk/mackay>

Initialize with:

$$\hat{\mathbf{w}}_0 = E[\mathbf{w}] \quad \mathbf{S}_{\mathbf{w}0} = \text{chol} \left\{ E[(\mathbf{w} - \hat{\mathbf{w}}_0)(\mathbf{w} - \hat{\mathbf{w}}_0)^T] \right\} \quad (30)$$

For $k \in \{1, \dots, \infty\}$,

Time update and sigma point calculation:

$$\hat{\mathbf{w}}_k^- = \hat{\mathbf{w}}_{k-1} \quad (31)$$

$$\mathbf{S}_{\mathbf{w}k}^- = \gamma^{-1/2} \mathbf{S}_{\mathbf{w}k-1} \quad (32)$$

$$\mathbf{W}_{k|k-1} = [\hat{\mathbf{w}}_k^- \quad \hat{\mathbf{w}}_k^- + \eta \mathbf{S}_{\mathbf{w}k}^- \quad \hat{\mathbf{w}}_k^- - \eta \mathbf{S}_{\mathbf{w}k}^-] \quad (33)$$

$$\mathcal{D}_{k|k-1} = \mathbf{G}[\mathbf{x}_k, \mathbf{W}_{k|k-1}] \quad (34)$$

$$\hat{\mathbf{d}}_k = \sum_{i=0}^{2L} W_i^{(m)} \mathcal{D}_{i,k|k-1} \quad (35)$$

Measurement update equations:

$$\mathbf{S}_{\mathbf{d}k} = \text{qr} \left\{ \left[\sqrt{W_1^{(c)}} [\mathcal{D}_{1:2L,k} - \hat{\mathbf{d}}_k] \quad \sqrt{\mathbf{R}^e} \right] \right\} \quad (36)$$

$$\mathbf{S}_{\mathbf{d}k} = \text{cholupdate} \left\{ \mathbf{S}_{\mathbf{d}k}, \mathcal{D}_{0,k} - \hat{\mathbf{d}}_k, W_0^{(c)} \right\} \quad (37)$$

$$\mathbf{P}_{\mathbf{w}k\mathbf{d}k} = \sum_{i=0}^{2L} W_i^{(c)} [W_{i,k|k-1} - \hat{\mathbf{w}}_k^-] [\mathcal{D}_{i,k|k-1} - \hat{\mathbf{d}}_k]^T \quad (38)$$

$$\mathcal{K}_k = (\mathbf{P}_{\mathbf{w}k\mathbf{d}k} / \mathbf{S}_{\mathbf{d}k}^T) / \mathbf{S}_{\mathbf{d}k} \quad (39)$$

$$\hat{\mathbf{w}}_k = \hat{\mathbf{w}}_k^- + \mathcal{K}_k (\mathbf{d}_k - \hat{\mathbf{d}}_k) \quad (40)$$

$$\mathbf{U} = \mathcal{K}_k \mathbf{S}_{\mathbf{d}k} \quad (41)$$

$$\mathbf{S}_{\mathbf{w}k} = \text{cholupdate} \left\{ \mathbf{S}_{\mathbf{w}k}^-, \mathbf{U}, -1 \right\} \quad (42)$$

where \mathbf{R}^e =measurement noise cov (this can be set to an arbitrary value, e.g., $.5\mathbf{I}$)

Algorithm 3.2: Square-Root UKF for parameter-estimation.

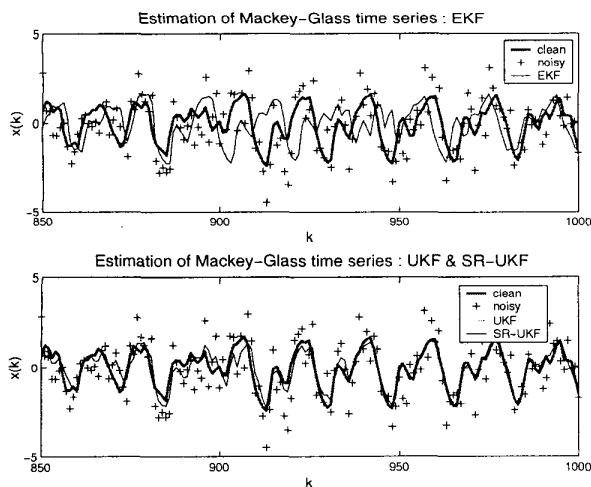


Figure 2: Estimation of the Mackey-Glass chaotic time-series with the EKF, UKF and SR-UKF.

5. CONCLUSIONS

The UKF consistently performs better than or equal to the well known EKF, with the added benefit of ease of implementation in

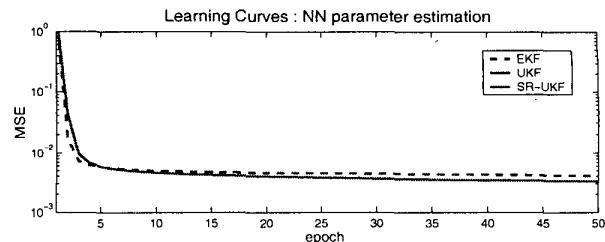


Figure 3: Learning curves for Mackay-Robot-Arm neural network parameter-estimation problem.

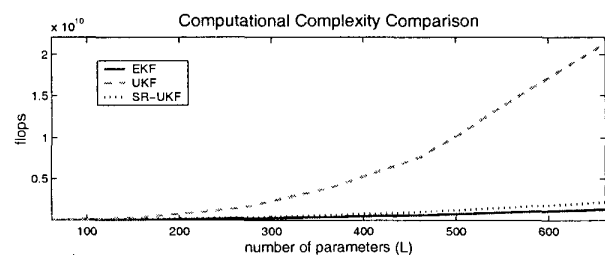


Figure 4: Computational complexity (flops/epoch) of EKF, UKF and SR-UKF for parameter-estimation (Mackay-Robot-Arm problem).

that no analytical derivatives (Jacobians or Hessians) need to be calculated. For state-estimation, the UKF and EKF have equal complexity and are in general $\mathcal{O}(L^3)$. In this paper, we introduced square-root forms of the UKF. The square-root UKF has better numerical properties and guarantees positive semi-definiteness of the underlying state covariance. In addition, for parameter-estimation an efficient $\mathcal{O}(L^2)$ implementation is possible for the square-root form, which is again of the same complexity as efficient EKF parameter-estimation implementations. In this light, the SR-UKF is the logical replacement for the EKF in all state and parameter-estimation applications.

6. REFERENCES

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